Math 2250-1

Mon. 12/5

§9.2-9.3 Classification of equilibrium solutions for autonomous systems of DE's, based on linearization; examples from population models.

recall:

$$x'(t) = F(x,y)$$
$$y'(t) = G(x,y)$$

if $P = \begin{bmatrix} x_* \\ y_* \end{bmatrix}$ is an equilibrium soln

and if we write $x(t) = x_* + u(t)$
$y(t) = y_* + v(t)$

and if we linearize the system about $\begin{bmatrix} x_* \\ y_* \end{bmatrix}$ then $\begin{bmatrix} u(t) \\ v(t) \end{bmatrix}$

almost satisfies

linearized system.

$$\begin{bmatrix} u'(t) \\ v'(t) \end{bmatrix} = \underbrace{\begin{bmatrix} \frac{\partial F}{\partial x}(P) & \frac{\partial F}{\partial y}(P) \\ \frac{\partial G}{\partial x}(P) & \frac{\partial G}{\partial y}(P) \end{bmatrix}}_{\text{"J"}} \begin{bmatrix} u \\ v \end{bmatrix}$$

↑ "J"

the Jacobian matrix of $\begin{bmatrix} F(x) \\ G(x) \end{bmatrix}$, evaluated at the equilibrium soln (aka "critical point") P.

We've been using the competition model

$$\frac{dx}{dt} = 14x - 2x^2 - xy \qquad \text{rabbits}$$

$$\frac{dy}{dt} = \underbrace{16y - 2y^2}_{\text{logistic}} \; \underbrace{-xy}_{\text{competition}} \qquad \text{squirrels}$$

to study linearization.... the idea is that one can <u>usually</u> deduce whether an equilibrium solution to the non-linear system is asymptotically stable or unstable, just based on the eigenvalues of the Jacobian matrix. Furthermore, in <u>most</u> <u>cases</u> the behavior of solutions to the non-linear system is mirrored almost exactly by the solutions to the linearized system, near the equilibrium point. (See page 3)

Exercise 1 Finish linearizing the rabbit-squirrel model at equilibrium points

(a) $(7,0)$

(b) $(0,0)$

(We did $(4,6)$ on Friday. You'll do $(0,8)$ in HW.)

$$x' = 14x - 2x^2 - xy$$
$$y' = 16y - 2y^2 - xy$$

Also classify the equilibria according to Figure 9.2.12 on next page.

Linearized sol's:

If λ is real
$$\vec{x}_H(t) = c_1 e^{\lambda_1 t}\vec{v}_1 + c_2 e^{\lambda_2 t}\vec{v}_2$$
(or if $\lambda_1 = \lambda_2$ is defective
$$\vec{x}_H(t) = c_1 e^{\lambda_1 t}\vec{v}_1 + c_2 e^{\lambda_1 t}(t\vec{v}_1 + \vec{u})$$
with $(A - \lambda_1 I)\vec{u} = \vec{v}_1$)

THEOREM 1 Stability of Linear Systems

Let λ_1 and λ_2 be the eigenvalues of the coefficient matrix **A** of the two-dimensional linear system
$$\frac{dx}{dt} = ax + by, \qquad \frac{dy}{dt} = cx + dy \tag{17}$$
with $ad - bc \neq 0$. Then the critical point (0, 0) is

1. Asymptotically stable if the real parts of λ_1 and λ_2 are both negative;
2. Stable but not asymptotically stable if the real parts of λ_1 and λ_2 are both zero (so that $\lambda_1, \lambda_2 = \pm qi$);
3. Unstable if either λ_1 or λ_2 has a positive real part.

$\lambda_1 < 0 < \lambda_2$

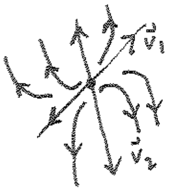
(unstable) $\boxed{\text{saddle}}$

THEOREM 2 Stability of Almost Linear Systems

Let λ_1 and λ_2 be the eigenvalues of the coefficient matrix of the linear system in (17) associated with the almost linear system in (18). Then

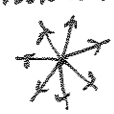
1. If $\lambda_1 = \lambda_2$ are equal real eigenvalues, then the critical point (0, 0) of (18) is either a node or a spiral point, and is asymptotically stable if $\lambda_1 = \lambda_2 < 0$, unstable if $\lambda_1 = \lambda_2 > 0$.
2. If λ_1 and λ_2 are pure imaginary, then (0, 0) is either a center or a spiral point, and may be either asymptotically stable, stable, or unstable.
3. Otherwise—that is, unless λ_1 and λ_2 are either real equal or pure imaginary—the critical point (0, 0) of the almost linear system in (18) is of the same type and stability as the critical point (0, 0) of the associated linear system in (17).

$0 < \lambda_1 < \lambda_2$



(unstable) $\boxed{\text{improper node}}$ (also nodal source)

$0 < \lambda_1 = \lambda_2$ non-defective



$\boxed{\text{proper node}}$

$\lambda_2 < \lambda_1 < 0$

$\boxed{\text{asympt. stable improper node}}$ (also nodal sink)

$\lambda_2 = \lambda_1 < 0$ non-defective a.s. proper node

Eigenvalues λ_1, λ_2 for the Linearized System	Type of Critical Point of the Almost Linear System
$\lambda_1 < \lambda_2 < 0$	Stable improper node
$\lambda_1 = \lambda_2 < 0$	Stable node or spiral point
$\lambda_1 < 0 < \lambda_2$	Unstable saddle point
$\lambda_1 = \lambda_2 > 0$	Unstable node or spiral point
$\lambda_1 > \lambda_2 > 0$	Unstable improper node
$\lambda_1, \lambda_2 = a \pm bi \quad (a < 0)$	Stable spiral point
$\lambda_1, \lambda_2 = a \pm bi \quad (a > 0)$	Unstable spiral point
$\lambda_1, \lambda_2 = \pm bi$	Stable or unstable, center or spiral point

URE 9.2.12. Classification of critical points of an almost linear system.

$\lambda = p \pm qi$ complex

for $\lambda = p + qi$ let $\vec{v} = \vec{a} + \vec{b}i$. Then
$$e^{\lambda t}\vec{v} = e^{(p+qi)t}(\vec{a} + \vec{b}i) = e^{pt}(\cos qt + i\sin qt)(\vec{a} + \vec{b}i)$$
$$= e^{pt}(\cos qt\,\vec{a} - \sin qt\,\vec{b}) + i e^{pt}(\cos qt\,\vec{b} + \sin qt\,\vec{a})$$
so $\vec{x}_H(t) = c_1 e^{pt}(\cos qt\,\vec{a} - \sin qt\,\vec{b}) + c_2 e^{pt}(\cos qt\,\vec{b} + \sin qt\,\vec{a})$
$$= e^{pt}\left[\vec{a} \,\middle|\, \vec{b}\right]\begin{bmatrix}\cos qt & \sin qt \\ -\sin qt & \cos qt\end{bmatrix}\left(\begin{bmatrix}c_1 \\ 0\end{bmatrix} + \begin{bmatrix}0 \\ c_2\end{bmatrix}\right)$$
$$= \underbrace{e^{pt}}_{\text{yields spiral if } p \neq 0}\underbrace{\left[\vec{a} \,\middle|\, \vec{b}\right]\underbrace{\begin{bmatrix}\cos qt & \sin qt \\ -\sin qt & \cos qt\end{bmatrix}}_{\text{circle}}\begin{bmatrix}c_1 \\ c_2\end{bmatrix}}_{\text{ellipse}}$$

$\lambda = p \pm qi$

$p < 0$

asy. stable $\boxed{\text{spiral pt}}$ $p < 0$ (spiral sink)

$p > 0$

unstable spiral pt $p > 0$ (spiral source)

$p = 0$

stable center for linear, indeterminate for non-linear

There's an interesting theorem related to the competition model (see text page 554).

For the system

$$x'(t) = a_1 x - b_1 x^2 - c_1 xy$$
$$y'(t) = a_2 y - b_2 y^2 - c_2 xy$$

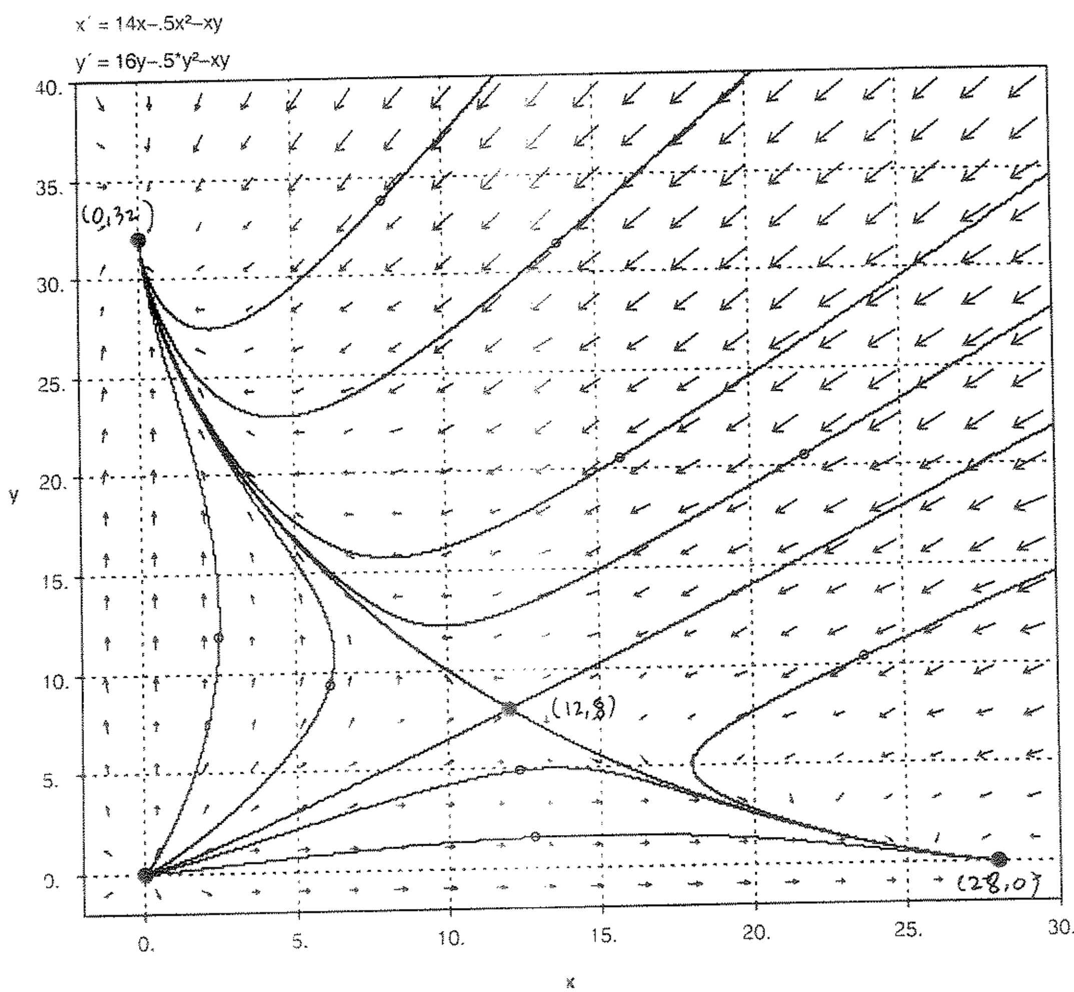
Suppose that there is an equilibrium soln in the first quadrant (i.e. (x_e, y_e) with $x_e > 0$, $y_e > 0$)

Then

1. if $c_1 c_2 < b_1 b_2$ (as in our example)
   (x_e, y_e) is a global attractor: As long as $x_0 > 0, y_0 > 0$, the solution to IVP $\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} \to \begin{bmatrix} x_e \\ y_e \end{bmatrix}$ as $t \to \infty$

2. if $c_1 c_2 > b_1 b_2$ (as in one of your HW problems), then
   (x_e, y_e) is a saddle, and for $x_0, y_0 > 0$ $\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} \to \begin{bmatrix} a_1/b_1 \\ 0 \end{bmatrix}$ or $\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} \to \begin{bmatrix} 0 \\ a_2/b_2 \end{bmatrix}$ with 100% probability!

example of ②. Notice how the stable and unstable orbits coming out of the saddle equilibrium divide the 1st quadrant into important regions...



Complex eigenvalues!

Exercise 2 a) Find equilibria & linearize at the one in the first quadrant, for this example of a logistic prey interacting with a predator

$$x'(t) = 10x - x^2 - 5xy \quad \leftarrow \text{prey (logistic)}$$
$$y'(t) = -5y + xy \quad \leftarrow \text{predator.}$$

b) knowing you've got a spiral point, check the tangent field at a few representative u,v locations to sketch a qualitatively accurate local picture near the 1st quadrant equilibrium solution