Math 2250-1
Fri 12/2

§9.2 Linearization near equilibria, for autonomous systems of 1^{st} order DE's.

On Wednesday we linearized the DE

$$\frac{dx}{dt} = 14x - 2x^2 - xy \qquad \text{rabbits}$$

$$\frac{dy}{dt} = \underbrace{16y - 2y^2}_{\text{logistic}} \underbrace{- xy}_{\text{competition}} \qquad \text{squirrels}$$

near the interesting equilibrium solution $\begin{bmatrix} 4 \\ 6 \end{bmatrix}$, as follows:

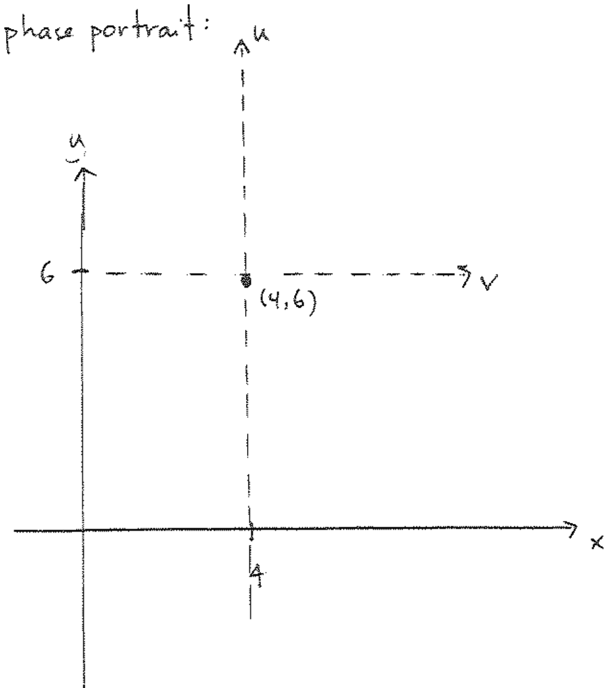
We wrote
$x = 4 + u$
$y = 6 + v$

so that if $\begin{bmatrix} x \\ y \end{bmatrix}$ is near $\begin{bmatrix} 4 \\ 6 \end{bmatrix}$, then $\begin{bmatrix} u \\ v \end{bmatrix}$ is near $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ (and vice versa)

$\rightarrow \begin{bmatrix} u \\ v \end{bmatrix}$ measures displacement from $\begin{bmatrix} 4 \\ 6 \end{bmatrix}$

$u' = x' = 14(4+u) - 2(4+u)^2 - (4+u)(6+v)$
$u' = 0 - 8u - 4v \quad -2u^2 - uv$ (after work!)
$v' = y' = 16(6+v) - 2(6+v)^2 - (4+u)(6+v)$
$v' = 0 - 6u - 12v$

phase portrait:



Exact DE's for u, v:

$$\begin{bmatrix} u' \\ v' \end{bmatrix} = \begin{bmatrix} -8 & -4 \\ -6 & -12 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} + \begin{bmatrix} -2u^2 - uv \\ -2v^2 - uv \end{bmatrix}$$

↑ linear piece.

↑ "error"; if $\left\| \begin{bmatrix} u \\ v \end{bmatrix} \right\| < \delta$ then $\|\text{error}\| < 6\delta^2$
↑ tiny ↑ tiny squared.

So, for $|u|, |v|$ tiny, exact solution should be close to linearized system

$$\boxed{\text{LS} \quad \begin{bmatrix} u' \\ v' \end{bmatrix} = \begin{bmatrix} -8 & -4 \\ -6 & -12 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}}$$

Exercise 1 Considering $\begin{bmatrix} u' \\ v' \end{bmatrix} = \begin{bmatrix} -8 & -4 \\ -6 & -12 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}$,

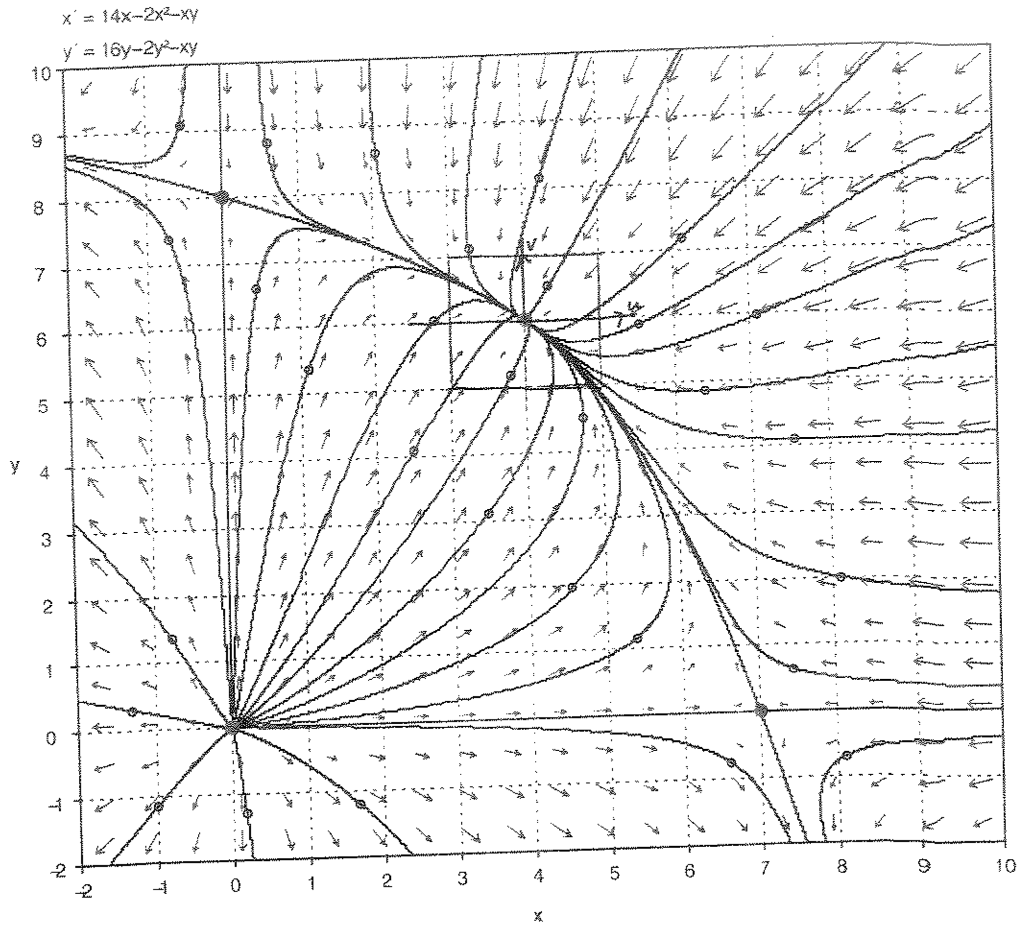
we have the following matrix eigendata

$\lambda_1 \cong -4.71 \qquad \lambda_2 \cong -15.3$

$\vec{v}_1 \cong \begin{bmatrix} .77 \\ -.63 \end{bmatrix} \qquad \vec{v}_2 \cong \begin{bmatrix} .49 \\ .89 \end{bmatrix}$

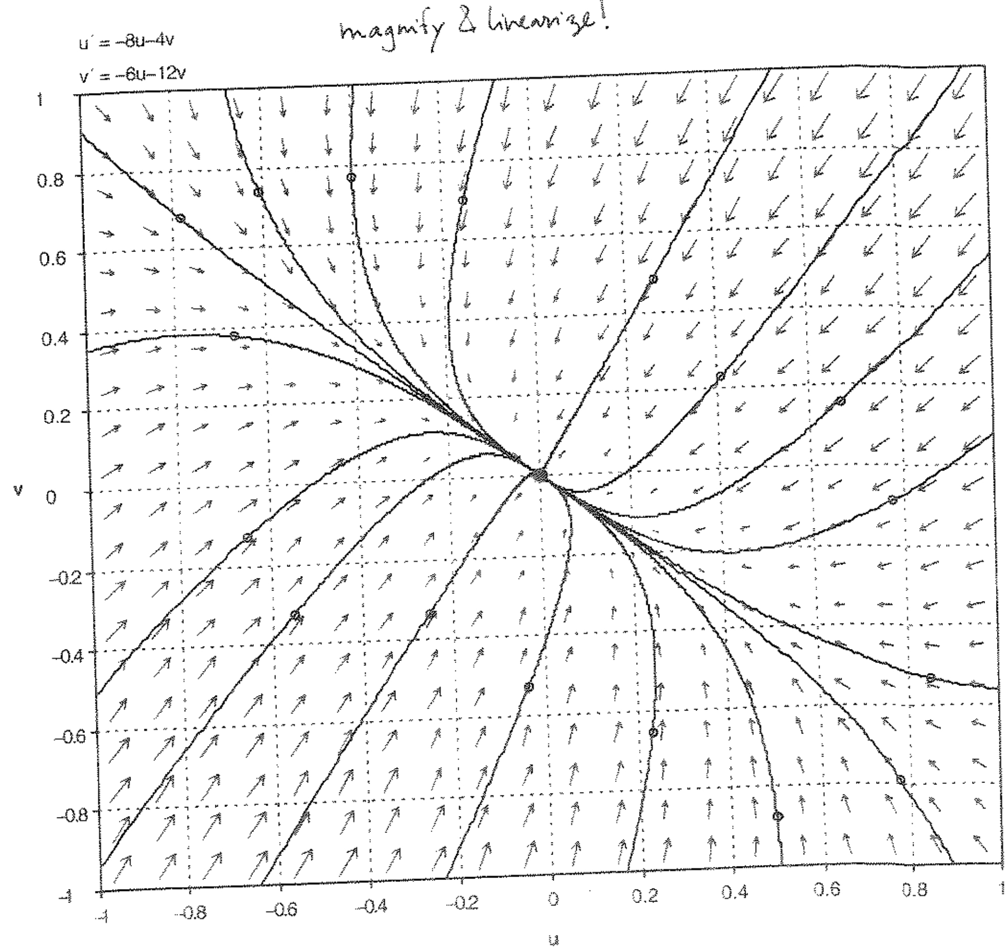
1a) Write the general soltn to the linearized system.

1b) Use the general solution (and the eigendata) to sketch a qualitatively accurate picture of the phase portrait for the linearized system. Compare with pplane output on next page.



What happens to rabbit-squirrel populations??

magnify & linearize!



$\boxed{\text{Shortcut to linearization:}}$ (works for systems of n DE's; illustrated for $n=2$)

Let
$$(1)\quad \begin{cases} x' = F(x,y) \\ y' = G(x,y) \end{cases}$$

$$F(x_*,y_*) = F(P) = 0$$
$$G(x_*,y_*) = G(P) = 0$$

write
$$x(t) = x_* + u(t)$$
$$y(t) = y_* + v(t)$$

we are interested in what happens for $\|(u,v)\|$ small.

error; $\frac{\varepsilon}{\|(u,v)\|} \to 0$ as $(u,v)\to(0,0)$ if partial derivs of F, G are continuous

$$x' = F(x_*+u, y_*+v) = \underbrace{F(x_*,y_*)}_{0} + F_x(x_*,y_*)u + F_y(x_*,y_*)v + \varepsilon_1(u,v)$$
$$y' = G(x_*+u, y_*+v) = \underbrace{G(x_*,y_*)}_{0} + G_x(-)u + G_y(-)v + \varepsilon_2(u,v)$$

$$u' = x' = F_x u + F_y v + \varepsilon_1(u,v)$$
$$v' = y' = G_x u + G_y v + \varepsilon_2(u,v)$$

where the partial derivs of F & G are evaluated at the equil. pt.

$$(2)\qquad \begin{bmatrix} u' \\ v' \end{bmatrix} = \underbrace{\begin{bmatrix} F_x & F_y \\ G_x & G_y \end{bmatrix}}_{\text{"}A\text{"}} \begin{bmatrix} u \\ v \end{bmatrix}$$

(Partial derivs in A are evaluated at (x^*,y^*))

this is the linearization of (1), at (x^*,y^*). the eigenvector data of A determines stability for the nonlinear system (1), in the non borderline cases.

(We use the same letters u, v as we did for the non-linear problem, even though the sol'n $\begin{bmatrix} u \\ v \end{bmatrix}$ to the linearized problem only approximate the translated sol'n $\begin{bmatrix} u \\ v \end{bmatrix}$ to the non-linear problem)

the matrix A is called the <u>Jacobian matrix</u> for $\vec{F}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{bmatrix} F(x,y) \\ G(x,y) \end{bmatrix}$, at $\begin{bmatrix} x_* \\ y_* \end{bmatrix}$

Exercise 2 Compute the linearizations of our rabbit-squirrel model on page 1 at the 4 equilibrium solns we found Wed.

2a • Verify we get the same linear system at $\begin{bmatrix}4\\6\end{bmatrix}$ that we got "the long way"

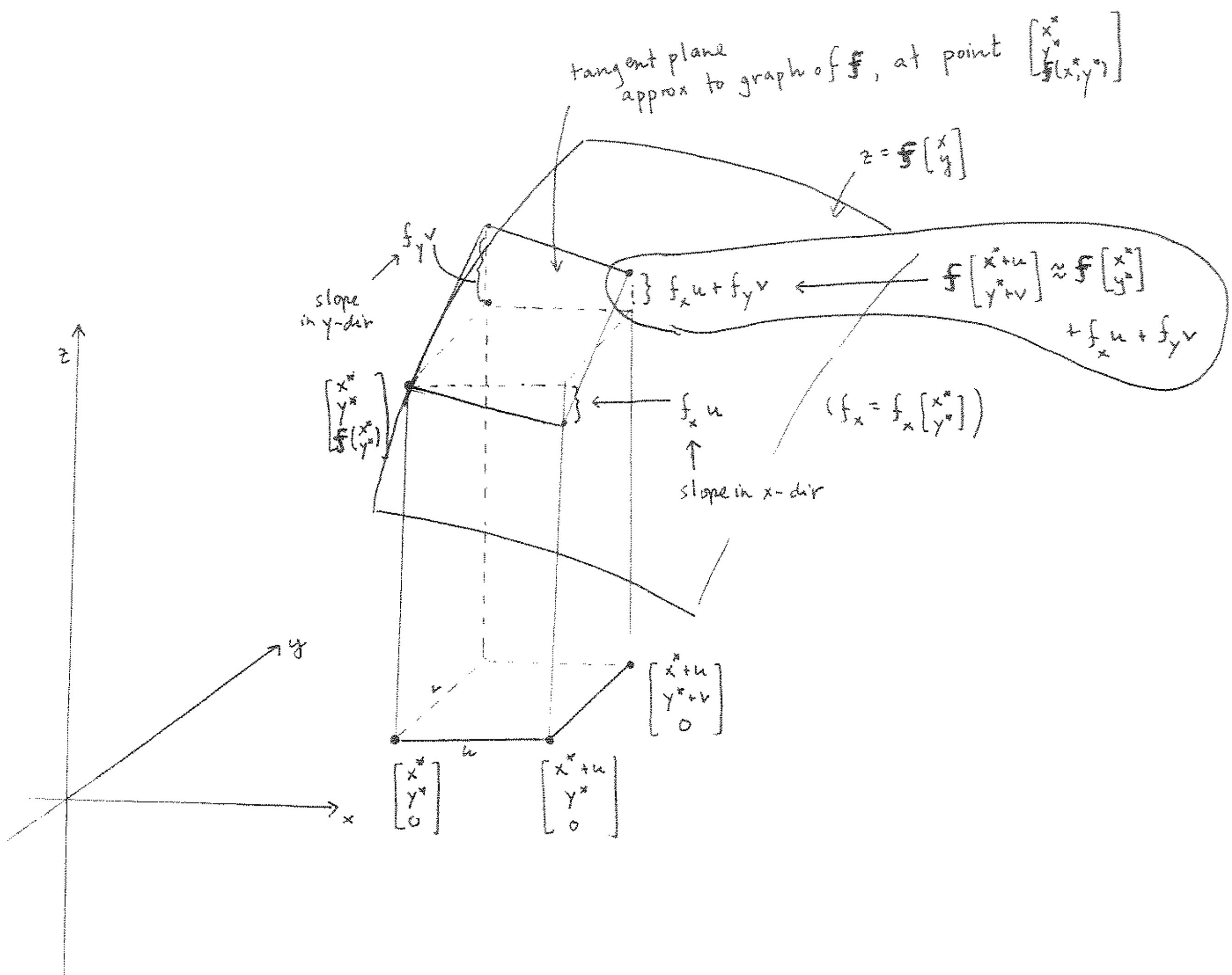
2b • Carry out the same eigenvector analysis at $\begin{bmatrix}0\\0\end{bmatrix}$, $\begin{bmatrix}7\\0\end{bmatrix}$ (and linearization) to explain the behavior of the non-linear problem at those equilibria.
(you'll do $\begin{bmatrix}0\\8\end{bmatrix}$ in HW.)

partial answer

$$J(x,y) = \begin{bmatrix} F_x & F_y \\ G_x & G_y \end{bmatrix} = \begin{bmatrix} 14-2x-y & -x \\ -y & 16-4y-x \end{bmatrix}$$

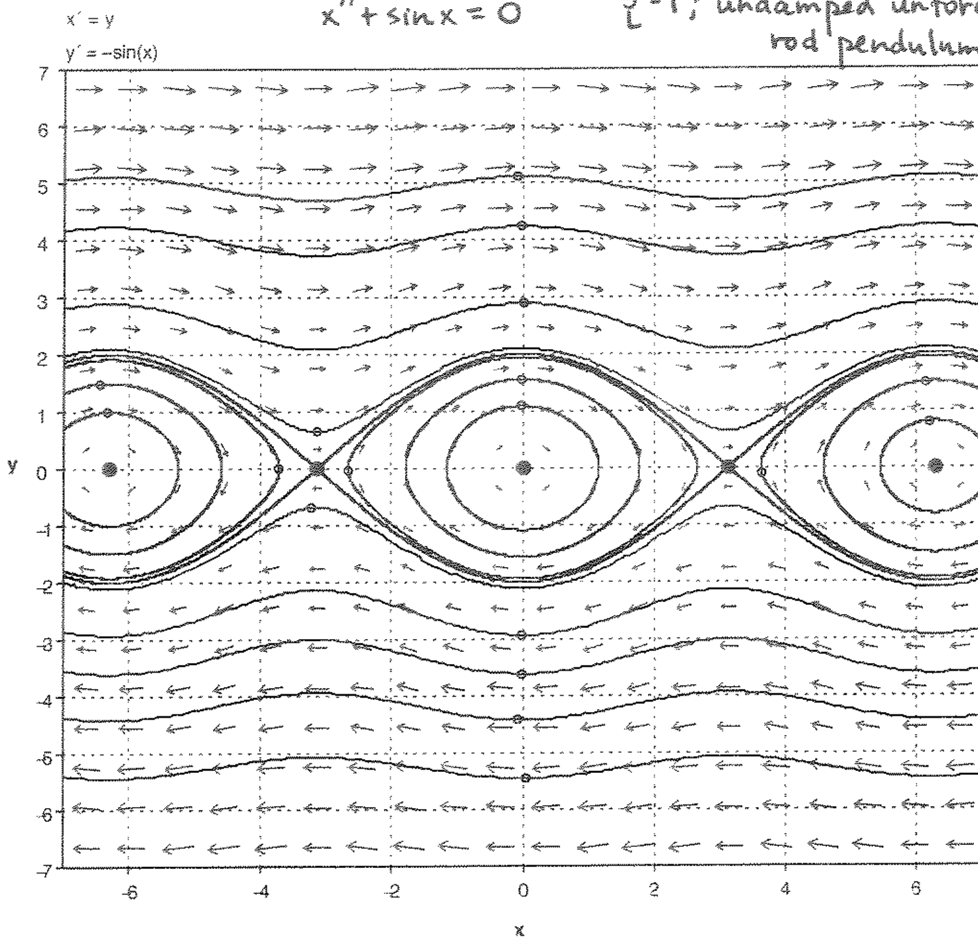
This schematic picture for the graph $z = f(x,y)$ might help you visualize the linearized approximation formula

$$f(x^*+u, y^*+v) \approx f(x^*, y^*) + f_x u + f_y v$$



next week! (§9.4)

$x'' + \sin x = 0$ $\quad$ $\frac{g}{L} = 1$; undamped unforced rigid rod pendulum!



$x'' + .2x' + \sin x = 0$ $\quad$ damping.