Math 2250
Wed 8/31 §1.5 & EP 3.7 (if time, begin §2.1)

We're discussing linear differential equations, how to solve them, and applications.

We'll do the example at the end of Tuesday's notes, but use exactly the same sort of DE to do a completely different (looking) electrical circuit example.

So that we can apply the same solution to both problems, first solve the IVP

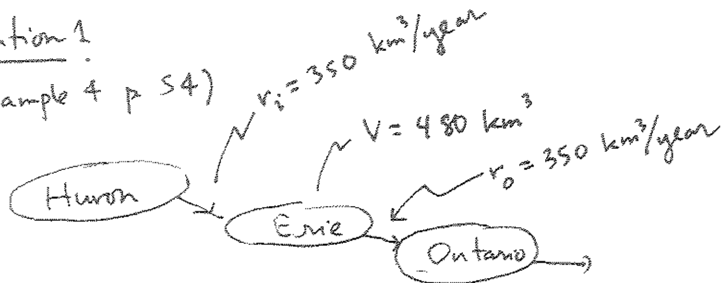
$$\text{IVP} \begin{cases} \dfrac{dx}{dt} + a x = b \\ x(0) = x_0 \end{cases} \qquad a, b, \text{ constants}$$

Use the linear DE algorithm!

soltn
$x(t) = \frac{b}{a} + (x_0 - b/a)e^{-at}$

application 1

(Example 4 p 54)



initial pollutant concentration
in Erie = $5c$, where c = concentration in Huron.

When will flushing cause concentration in Erie to decrease to $2c$?

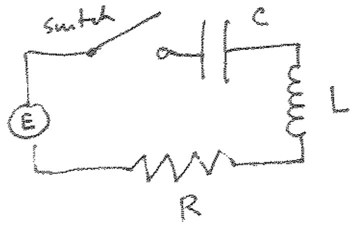
- Since $r_i = r_o$, volume of Erie is constant V
- $\frac{dx}{dt} = r_i c_i - r_o c_o$. Call common value $r = r_i = r_o$

$$\begin{cases} \frac{dx}{dt} = rc - \frac{r}{V}x \\ x(0) = ? \end{cases}$$

- Use page 1 soltn to write down $x(t)$ in this case
- Use soltn to solve problem

ans $a = \frac{r}{V}$, $b = rc$,
$x(t) = cV + 4cVe^{-\frac{r}{V}t}$
$t \cong 1.90$ years.

EP 3.7 Electrical circuits, analog.



Voltage E (called electromotive force, but really the units are energy/unit charg)

circuit elt	Voltage drop	units for component
Inductor	$L \frac{dI}{dt}$	L Henries (H)
Resistor	RI	R Ohms (Ω)
Capacitor	$\frac{1}{C} Q(t)$	C Farads (F)

$Q(t)$ = net charge accumulation (Coulombs)
$I(t) = Q'(t)$ = current (Amperes) (moving around loop)

Kirchoff's Law

the sum of the voltage drops around any closed circuit loop equals the applied voltage E

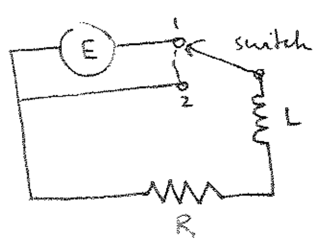
(this says that a test particle traversing any closed loop returns with the same potential energy level it started with)

$$L Q''(t) + R Q'(t) + \frac{1}{C} Q(t) = E(t)$$

$$D_t: \quad L I''(t) + R I'(t) + \frac{1}{C} I(t) = E'(t)$$

If no capacitor $\rightarrow$ 1st order DE for $I(t)$: $L I' + R I = E(t)$
If no inductor $\rightarrow$ 1st order DE for $Q(t)$: $R Q' + \frac{1}{C} Q = E$
or for $I(t)$: $R I' + \frac{1}{C} I = E'$

Example Set up & solve IVP for this circuit and problem; for $I(t)$. What is the $\lim_{t \to \infty} I(t)$?
(application 2)



Suppose $L = 10\,H$, $R = 20\,\Omega$, $E = 100\,V$
Suppose switch has been in position 2 for a long time so there is no current flowing
@ $t = 0$ switch is moved to position 1

IVP $\Big\{$

soln
$I(t) = 5 - 5e^{-2t}$

Chapter 2: 2.1–2.3 improved models for familiar applications
2.4–2.6 numerical methods

§2.1 Improved population models

Let $P(t)$ = population (e.g. # people, # fish, etc.)
$B(t)$ = birth rate (e.g. people/year)
$D(t)$ = death rate (e.g. people/year)

$\beta(t) = \dfrac{B(t)}{P(t)}$ (people/year)/people e.g. .1 person/year per person.
fertility rate

$\delta(t) = \dfrac{D(t)}{P(t)}$ morbidity rate

Then

$$\frac{dP}{dt} = B(t) - D(t)$$

$$\boxed{\frac{\left(\frac{dP}{dt}\right)}{P} = \beta(t) - \delta(t)}$$

model 1: $\beta(t) \equiv \beta$ const, $\delta(t) \equiv \delta$ const $\Rightarrow \dfrac{dP}{dt} = (\beta-\delta)P = kP$

either exponential growth or decay, depending on sign of k
... we know this DE!

model 2: $\beta = \beta_0 - \beta_1 P$ (β_0, β_1 const)
$\delta = \delta_0 + \delta_1 P$

$\beta_1 > 0$ "sophisticated" population
$\delta_1 > 0$ ~ finite resources, disease, fighting

So

$$\begin{aligned}\frac{dP}{dt} &= (\beta-\delta)P \\ &= [(\beta_0 - \beta_1 P) - (\delta_0 + \delta_1 P)]\,P \\ &= [(\beta_0 - \delta_0) - (\beta_1 + \delta_1)P]\,P\end{aligned}$$

$$\boxed{\begin{aligned}\frac{dP}{dt} &= kP(M-P) \\ \frac{dP}{dt} &= aP - bP^2\end{aligned}}$$

$k = \beta_1 + \delta_1$, $M = \dfrac{\beta_0 - \delta_0}{\beta_1 + \delta_1}$

$a = kM$, $b = k$ $\left(\begin{aligned} k &= b \\ M &= a/b\end{aligned}\right)$

Logistic eqtn

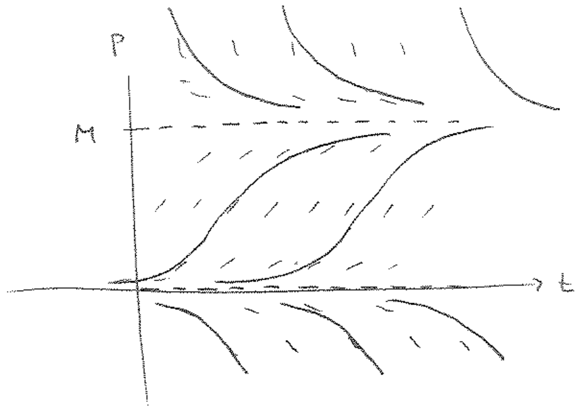
(There is a model 3 we'll discuss later, for which $\beta = \beta_0 + \beta_1 P$ ("alligators") which can lead to the "doomsday-extinction" de, $\dfrac{dP}{dt} = -aP + bP^2$.)

What does the slope field predict for sol'ns to

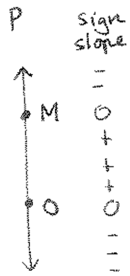
$$\text{IVP}\begin{cases}\dfrac{dP}{dt} = kP(M-P) & (k, M>0)\\ P(0)=P_0\end{cases}$$

notice isoclines are horizontal lines in the (t,P) plane:

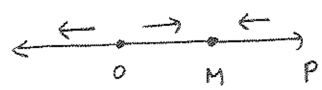
$P=0 \Rightarrow m=0$
$P=M \Rightarrow m=0$



slope field picture



leads to phase portrait:



$P=0$ and $P=M$ are called <u>equilibrium solutions</u> (because they are constant in time)

- It appears that any IVP sol'n $P(t)$ with $P(0)=P_0>0$ will converge to $P=M$.

$\boxed{M := \text{carrying capacity.}}$

$\boxed{\text{Solve the logistic IVP by separating variables}}$ (Use back of page if necessary!)
(Then analyze solution to verify slope field & phase portrait predictions)

Sol: $P(t) = \dfrac{M}{\left(\frac{M-P_0}{P_0}\right)e^{-Mkt}+1}$