Math 2250-1
Tues 30 Aug

§1.5 Linear DE's cont'd. (We'll begin class with p. 4-6 Monday notes)

Recall a 1st-order DE is linear if it can be written in the form

$$y' + P(x)y = Q(x) \qquad\qquad y = y(x)$$

Solve via integrating factor

$$e^{\int P(x)dx}\left[y' + P(x)y\right] = e^{\int P(x)dx} Q(x)$$

$$\left[e^{\int P(x)dx} y\right]' = e^{\int P(x)dx} Q(x)$$

write $\bar{P}(x) = \int P(x)dx$ for our antideriv. choice

$$\bar{P}' = P$$

$$\left[e^{\bar{P}(x)} y\right]' = e^{\bar{P}(x)} Q(x)$$

antidiff

$$e^{\bar{P}(x)} y(x) = \int e^{\bar{P}(x)} Q(x)dx + C.$$

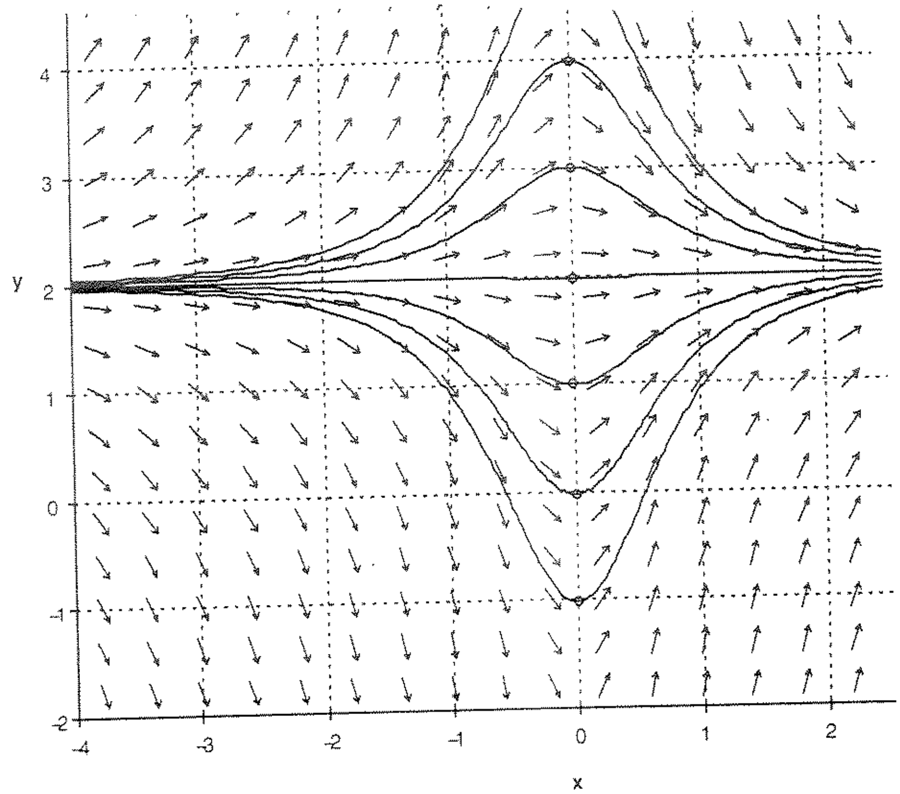
$$y(x) = e^{-\bar{P}(x)} \int e^{\bar{P}(x)} Q(x)dx + Ce^{-\bar{P}(x)}$$

adjust C to solve any IVP

Exercise 1 : p 50-51. Solve

$$\frac{dy}{dx} = \frac{6x - 3xy}{x^2+1}$$

be rewriting it as a linear DE.
(Notice it is also separable)



ans $y = 2 + C(x^2+1)^{-3/2}$

Notice, for DE

IVP $\begin{cases} \frac{dy}{dx} + P(x)y = Q(x) \\ y(x_0) = x_0 \end{cases}$

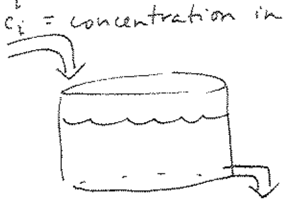
If $P(x), Q(x)$ are continuous on the interval I, with $x_0 \in I$,
then $\int P(x)dx$ exists (e.g. $\int_{x_0}^{x} P(r)dr$ is one.)
and $\int Q(x) e^{\int P(x)dx} dx$ exists, so our page 1 formula
always yields a soltn. $\exists!$ Theorem guarantees its unique on
the entire interval ($\frac{\partial f}{\partial y} = -P(x)$ is cont.)

- for linear DE solution exists & is unique on entire interval, so in particular
    - no vertical asymptotes
    - different solution curves can't intersect.
    - soltn doesn't just "stop" at a vertical point.

this is a theorem! p. 51

• Example: <u>mixing problems</u> (p. 51) applies also to pharmacology & environment

r_i = rate in ℓ/s
c_i = concentration in gm/ℓ



r_o = rate out ℓ/s
c_o = concentration out g/ℓ

$V(t)$ = volume in tank at time t (ℓ)
$x(t)$ = amount of solute in tank (gm)

$c(t) = \dfrac{x(t)}{V(t)}$ $\frac{g}{\ell}$ (average) concentration in tank

Assume mixture is uniform so $c(t)$ is spatially constant

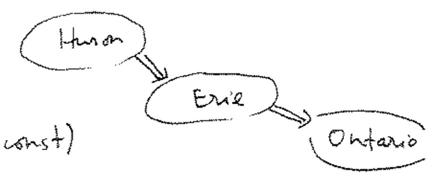
$$\frac{dV}{dt} = r_i - r_o \quad \ell/s \longrightarrow V(t) = V_0 + \int_0^t r_i(s) - r_o(s)\,ds \qquad \S 1.2$$

$$\frac{dx}{dt} = r_i c_i - r_o c_o \qquad \frac{\ell}{s}\frac{g}{\ell} \quad g/s$$

$$= r_i c_i - r_o \frac{x(t)}{V(t)} \qquad \text{by "well mixed" model}$$

$$\boxed{\frac{dx}{dt} + \underbrace{\frac{r_o}{V(t)}}_{P(t)} x(t) = \underbrace{r_i c_i}_{Q(t)}}$$

<u>Example 4 p. 52</u> Lake Erie pollution



$V(t) = 480 \text{ km}^3$
$r = r_i = r_o = 350 \text{ km}^3/\text{year}$ (so V is const)
$c_i = c$ = pollutant conc. of lake Huron
$x_0 = (5c)V$ (read problem).

When will Erie pollution concentration have decreased to twice that of Huron?

i.e. solve, and analyze

$$\begin{cases} \frac{dx}{dt} = r_i c_i - r_o c_o \\ x(0) = 5cV \end{cases}$$

$$\frac{dx}{dt} = rc - r\frac{x}{V}$$

for fun, notice this DE is also separable. Can you solve it that way?

$$\frac{dx}{dt} + \frac{r}{V}x = rc$$

$$\left(e^{\frac{r}{V}t} x\right)' = rc\, e^{\frac{r}{V}t}$$

$$e^{\frac{r}{V}t} x = rc\frac{V}{r} e^{\frac{r}{V}t} + C$$

$$x = cV + Ce^{-\frac{r}{V}t}$$

$$x(0) = 5cV \Rightarrow C = 4cV$$

$$x(t) = cV + 4cVe^{-\frac{r}{V}t}$$

set $x(T) = 2cV = cV + 4cVe^{-\frac{r}{V}T}$

$$.25 = e^{-\frac{r}{V}T}$$

$$-\frac{r}{V}T = \ln(.25)$$

$$T = -\ln(.25)\frac{V}{r}$$

$$= \ln(4)\frac{480}{350} \text{ years}$$

$$\boxed{\approx 1.90 \text{ years}}$$