Math 2250-1

Monday Aug 29 §1.4-1.5

Separable DE modeling/experiment day + intro to linear DE's
&
Introduction to MAPLE.

Maple introduction sessions: LCB 115

Mon 8/29 3-3:50 pm
Tues 8/30 11:50-12:40
Wed 8/31 9:40-10:30
2-2:50

Fri 9/2 10:45-11:35
3-3:50

In §1.4 text & hw there are exponential growth/decay + Newton's law of cooling examples. These should be review for you, especially after last week's examples.

Here's a model that might be new to you:

Toricelli's Law for draining tanks

Let a tank be draining water. Let $y(t)$ be the water height above the drain. Then the speed v with which the water leaves the tank is given by

$$\boxed{v = \sqrt{2gy}}$$

g = acceleration of gravity.

derivation neglecting friction yields a conservative system for which

$$KE + PE = const$$

in a small time interval dt there is a (negative) change of water mass in the tank, dM.

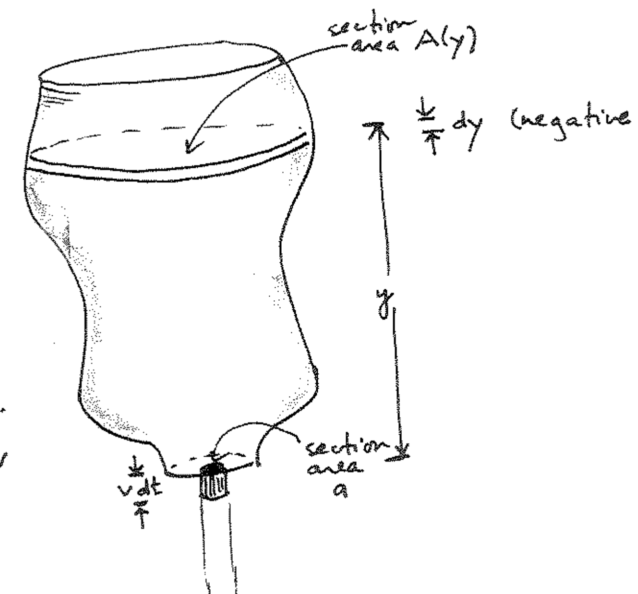
An equal mass has drained from the drain at speed v.

By conservation of energy,

loss in PE = gain in KE

$$\boxed{\cancel{|dM|}\, g\, y = \tfrac{1}{2} \cancel{|dM|}\, v^2}\ {}^*$$

thus $\boxed{v = \sqrt{2gy}}$



* if $a \ll A$ we may neglect the KE of the slow moving water inside the vessel.

Toricelli's Law yields a separable DE for height $y(t)$:

$dM = \rho A(y)\, dy$ ρ = density (1 g/cc)

$dM = \rho a(-v\, dt)$ (measured at bottom)

$A(y)\, dy = -a v\, dt$

$A(y)\, dy = -a\sqrt{2gy}\, dt$ Toricelli

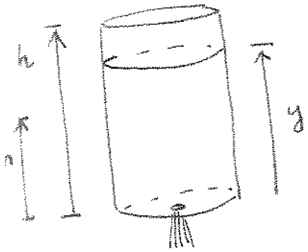
$$\boxed{A(y)\frac{dy}{dt} = -k\sqrt{y}}$$

$k = a\sqrt{2g}$

experiment!!

Exercise 1 Cylindrical cistern: $A(y) = A$ const.

$$\frac{dy}{dt} = -k y^{1/2} \quad \text{(different } k\text{)}.$$



Let T_μ be the amount of time it takes the cistern to empty from full height h, to height μh $(0 \le \mu \le 1)$

Show the time it takes to empty the tank is given by

$$\boxed{T_0 = \frac{T_\mu}{1-\mu^{1/2}}}$$

use separation of variables!

Nalgene bottle experiment:

I marked off the bottle so that we can use $\mu = .5$

So if we time how long it takes to empty half the height, and call it $T_{1/2}$, then the total time estimate will be

$$T_0 = \frac{1}{1-\sqrt{.5}} T_{.5} \approx (3.41) T_{.5}$$

Experiment

$T_{1/2} =$

$(3.41) T_{1/2} =$

$T_0 =$

Math 2250–1
Introduction to MAPLE
Monday August 29

This is a Maple 15 document. As you can see it's a mixture of text and mathematics. I've created it as a "Maple Document", as opposed to "Maple Worksheet." I will use the menu bar at the top of Maple to keep the unbracketed fields as text (or copied and pasted Maple work), and use the brackets with ">" prompts to do math.

1a) Toricelli Experiment: Here I'll just use Maple as a calculator which records computations into an easily usable file. These are numbers I got in my office today. We can see how they compare to what we do in class.

Model:

> $\dfrac{1}{1-\mathrm{sqrt}(.5)}$;

$$3.414213563 \tag{1}$$

> $Thalf := 35.;$
$Tpredict := 3.414 \cdot Thalf;$

$$Thalf := 35.$$
$$Tpredict := 119.490 \tag{2}$$

> $Tact := 60 + 53;$

$$Tact := 113 \tag{3}$$

> $\dfrac{Tpredict}{Tact}$; *#only accurate to within 6%:*

$$1.057433628 \tag{4}$$

>

Notice that the experiment is close, but doesn't exactly match the prediction. Aside from experimental inaccuracies, can you think of places in which the model itself is inexact (there are several)?

1b) Here's a Maple picture of the direction field for the differential equation. Explain where uniqueness for the IVP fails, and what it means in terms of the experiment we just did:

> *with(DETools) : #differential equations package*

> *deqtn := diff(y(x), x) =-sqrt(y(x)) : #this is the DE*
> *dsolve({deqtn, y(0) = 1}); #solution!*

$$y(x) = \frac{1}{4}x^2 - x + 1 \tag{5}$$

> $factor\left(\frac{1}{4}x^2 - x + 1\right)$; *#of course, you could factor by hand too*

$$\frac{1}{4}(x-2)^2 \tag{6}$$

> *DEplot(deqtn, y(x), 0..3, {[y(0) = 0], [y(0) = 1.],*
> *[y(0) = 2.], [y(0) = 0.5], [y(0) = 1.5]},*
> *y = 0..2, arrows = line, color = black, linecolor = black,*
> *dirgrid = [30, 30], stepsize = .1, title = 'Toricelli'); #Maple's not as*
> *#cool as "dfield", but you can still draw slope fields and solution*
> *#trajectories*

```
Warning, plot may be incomplete, the following errors(s) were issued:
   cannot evaluate the solution further right of 1.4141924, probably a
singularity
Warning, plot may be incomplete, the following errors(s) were issued:
   cannot evaluate the solution further right of 1.9999776, probably a
singularity
Warning, plot may be incomplete, the following errors(s) were issued:
   cannot evaluate the solution further right of 2.4494671, probably a
singularity
Warning, plot may be incomplete, the following errors(s) were issued:
   cannot evaluate the solution further right of 2.8283979, probably a
singularity
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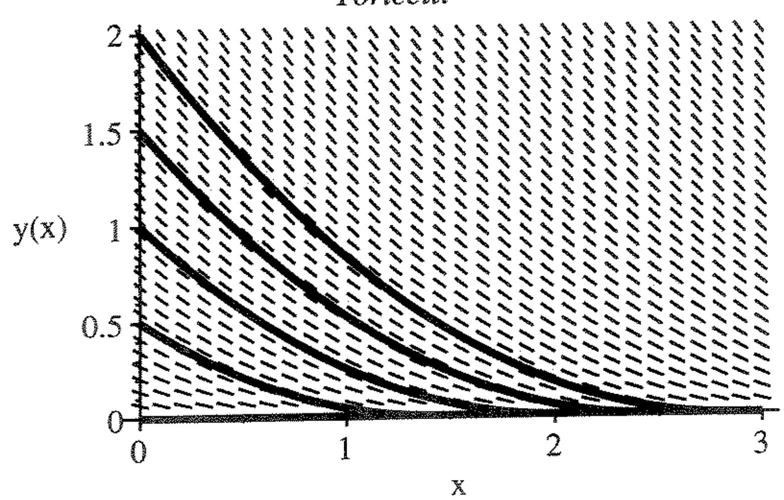
Toricelli



.5 Linear 1^{st} order DE's.

$$(1) \qquad \frac{dy}{dx} + P(x)y = Q(x)$$

Notice the left side of this eqtn, $L(y) := y' + P(x)y$ is <u>linear</u> :

$$L(y_1 + y_2) = L(y_1) + L(y_2) \qquad (y_1, y_2 \text{ diffble})$$
$$L(cy) = cL(y) \qquad (c \text{ a const, } y \text{ diffble})$$

Solution method is to multiply
both sides by a non-zero fcn ("integrating factor")
so that we can antidifferentiate wrt x to deduce $y(x)$:

Eqtn (1) is equiv. to

$$(2) \qquad e^{\int P(x)dx}\left(y' + P(x)y\right) = e^{\int P(x)dx}\,Q(x)$$

where $\int P(x)dx$ is any particular antideriv. of $P(x)$

equiv to

$$(3) \qquad \frac{d}{dx}\left[e^{\int P(x)dx}\,y\right] = \underline{e^{\int P(x)dx}\,Q(x)}$$

or (antidifferentiating)

$$(4) \qquad e^{\int P(x)dx}\,y = \int [\qquad] \, dx$$

; divide by $e^{\int P(x)dx}$ to get $y(x)$

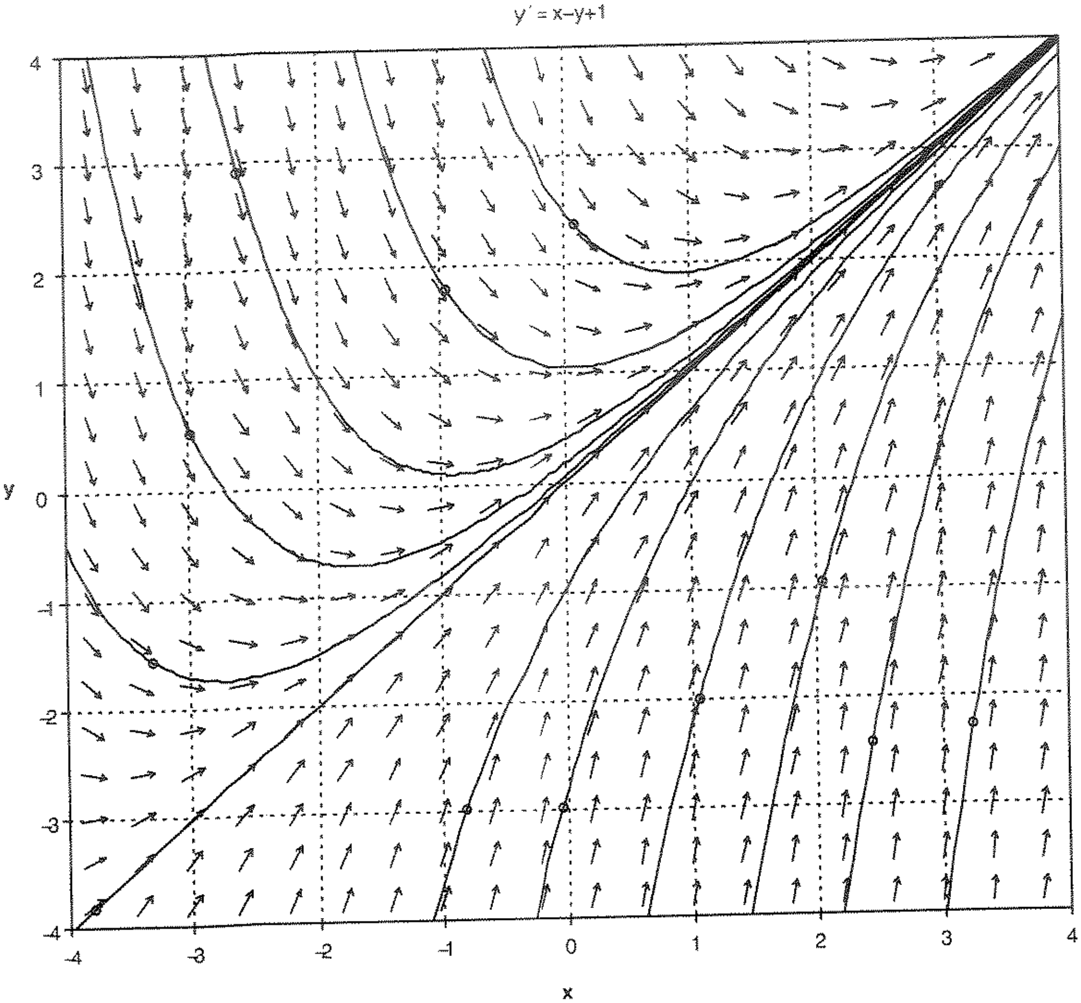
<u>Exercise 2a</u> Consider the DE

$$y' = x - y + 1$$

Rewrite it in linear form & solve as above

<u>ans</u> $y = x + Ce^{-x}$

2b) Are your answers to 2a) consistent with the slope field for
$y' = x - y + 1$?



Exercise 3 (p. 51) Solve

$$(x^2+1)\frac{dy}{dx} + 3xy = 6x$$

ans $y = 2 + C(x^2+1)^{-3/2}$