Name SOLUTIONS

I.D. number

Math 2250-1
FINAL EXAM
Dec 16, 2008

This exam is closed-book and closed-note. You may use a scientific calculator, but not one which is capable of graphing or of solving differential or linear algebra equations. Laplace Transform Tables are included with this exam. **In order to receive full or partial credit on any problem, you must show all of your work and justify your conclusions.** This exam counts for 30% of your course grade. It has been written so that there are 200 points possible, however, and the point values for each problem are indicated in the right-hand margin. **Good Luck!**

problem	score	possible
1	_______	35
2	_______	30
3	_______	20
4	_______	10
5	_______	30
6	_______	35
7	_______	40
Total	_______	200

1a) A motorboat containing the pilot has total mass of 360 kilograms, and its motor is able to provide 120 Newtons of thrust. However, when the boat is in motion, drag from the water produces a force of 6 newtons for each meter/sec of boat velocity. Use Newton's law to explain why (while the motor is on) the boat velocity satisfies the differential equation

$$\frac{dv}{dt} = \frac{1}{3} - \frac{v}{60}$$

(5 points)

$m\frac{dv}{dt}$ = net forces

$$360\frac{dv}{dt} = \underbrace{120}_{\text{thrust}} - \underbrace{6v}_{\text{drag.}} \qquad \div 360 \Rightarrow \boxed{\frac{dv}{dt} = \frac{1}{3} - \frac{v}{60}}$$

1b) What is the equilibrium solution for the velocity v(t)? Is this solution stable or unstable? Explain with a phase diagram.

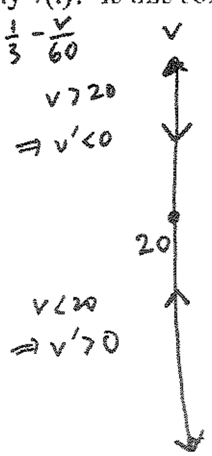
(5 points)

$$\frac{1}{3} - \frac{v}{60} = 0$$

$$\frac{v}{60} = \frac{1}{3}$$

$$v = \frac{60}{3} = 20 \text{ m/sec.}$$

$v' = \frac{1}{3} - \frac{v}{60}$



So v = 20 is stable (asymptotically stable!)

1c) Solve the initial value problem for the boat's velocity, assuming the boat starts at rest. Use the integrating-factor method we learned in Chapter 1, for first order linear differential equations.

(10 points)

$$\begin{cases} v' = \frac{1}{3} - \frac{v}{60} \\ v(0) = 0 \end{cases}$$

$$v' + \frac{1}{60} v = \frac{1}{3}$$

$$e^{\frac{1}{60}t}\left(v' + \frac{1}{60}v\right) = \frac{1}{3}e^{\frac{1}{60}t}$$

$$\left(e^{\frac{1}{60}t} v\right)'$$

integrate: $e^{\frac{1}{60}t} v = \int \frac{1}{3} e^{\frac{1}{60}t}\, dt$

$$= \frac{\frac{1}{3}}{1/60} e^{1/60\, t} + c$$

$$= 20\, e^{\frac{1}{60}t} + c$$

$\div e^{\frac{1}{60}t}$: $v = 20 + Ce^{-\frac{1}{60}t}$

$v(0) = 0 = 20 + C \Rightarrow C = -20$

$$\boxed{v = 20 - 20e^{-\frac{1}{60}t}}$$

1d) Resolve the same IVP, this time using the algorithm for separable differential equations.

(10 points)

$$v' = \frac{1}{3} - \frac{v}{60}$$

$$\frac{dv}{dt} = -\frac{1}{60}(v - 20)$$

$$\frac{dv}{v - 20} = -\frac{1}{60}\, dt$$

$\int$: $\ln|v - 20| = -\frac{1}{60}t + C_1$

exp: $|v - 20| = e^{C_1} e^{-\frac{1}{60}t}$

remove abs: $v - 20 = Ce^{-\frac{1}{60}t}$

$v = 20 + Ce^{-\frac{1}{60}t}$

$v(0) = 0$, so $C = -20$ (as above)

$$\boxed{v = 20 - 20e^{-\frac{1}{60}t}}$$

1e) If the boat starts at rest, how long does it take to reach 75% of its terminal velocity?

(5 points)

terminal vel = 20 m/sec.

75% (20) = 15 m/sec.

$$15 = 20 - 20e^{-\frac{1}{60}t}$$

$$-5 = -20e^{-\frac{1}{60}t}$$

$$\frac{1}{4} = e^{-\frac{1}{60}t}$$

ln: $-\ln 4 = -\frac{1}{60}t \Rightarrow \boxed{t = 60 \ln 4 \text{ sec.}}$

2) Consider the following differential equation, which could arise as an unforced forced spring problem:

$$x''(t) + 5\,x'(t) + 6\,x(t) = 0.$$

2a) Use the characteristic polynomial to find the general solution to this differential equation. What kind of damping governs this DE?

(8 points)

$p(r) = r^2 + 5r + 6 = (r+3)(r+2)$ so roots $r = -3, -2$

$\vec{x}_H(t) = c_1 e^{-3t} + c_2 e^{-2t}$

over damped

(real negative roots, soln passes equil $(x=0)$ at most once

2b) Use your work in (2a) to solve the initial value problem

$$x''(t) + 5\,x'(t) + 6\,x(t) = 0$$
$$x(0) = 1$$
$$x'(0) = -1$$

(12 points)

$x(t) = c_1 e^{-3t} + c_2 e^{-2t}$

$x' = -3c_1 e^{-3t} - 2c_2 e^{-2t}$

$x(0) = c_1 + c_2 = 1$

$x'(0) = -3c_1 - 2c_2 = -1$

$$\begin{bmatrix} 1 & 1 \\ -3 & -2 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \frac{1}{1}\begin{bmatrix} -2 & -1 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

so $\boxed{x(t) = -e^{-3t} + 2e^{-2t}}$

2c) Resolve the initial value problem above, using Laplace transforms.

(10 points)

$s^2 X(s) - s + 1 + 5\,(sX(s) - 1) + 6X(s) = 0$

$X(s)\,(s^2 + 5s + 6) = s + 4$

$X(s) = \dfrac{s+4}{s^2+5s+6} = \dfrac{A}{s+2} + \dfrac{B}{s+3} = \dfrac{A(s+3) + B(s+2)}{(s+2)(s+3)}$

$s + 4 = A(s+3) + B(s+2)$

@ $s = -3$: $\quad 1 = -B \Rightarrow B = -1$

@ $s = -2$: $\quad 2 = A$

so $X(s) = \dfrac{2}{s+2} - \dfrac{1}{s+3}$

So $\boxed{x(t) = 2e^{-2t} - e^{-3t}}$

3) Consider the first order system of differential equations

$$\begin{bmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -6 & -5 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}.$$

3a) Find the general solution to this system, using the eigenvalue/eigenvector method.

(15 points)

$$\begin{vmatrix} -\lambda & 1 \\ -6 & -5-\lambda \end{vmatrix} = \lambda^2 + 5\lambda + 6 = (\lambda+3)(\lambda+2) = 0$$

$$\lambda = -3, -2$$

$\lambda = -3$:

$$\left[\begin{array}{cc|c} 3 & 1 & 0 \\ -6 & -2 & 0 \end{array}\right] \quad \left[\begin{array}{cc|c} 3 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right] \qquad \vec{v} = \begin{bmatrix} 1 \\ -3 \end{bmatrix}$$

$\lambda = -2$

$$\left[\begin{array}{cc|c} 2 & 1 & 0 \\ -6 & -3 & 0 \end{array}\right] \quad \left[\begin{array}{cc|c} 2 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right] \qquad \vec{v} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

$$\boxed{\vec{x}(t) = c_1 e^{-3t} \begin{bmatrix} 1 \\ -3 \end{bmatrix} + c_2 e^{-2t} \begin{bmatrix} 1 \\ -2 \end{bmatrix}}$$

3b) Explain how this system is related to the second order differential equation in problem (2), and how you could have deduced the general solution in (3a) directly from the one you found in (2b).

(5 points)

If we convert (2a): $x'' + 5x' + 6x = 0$ into an equivalent first order system, by

$x = x$ then $x' = y$

$y = x'$ $\quad y' \; (= x'') = -5x' - 6x = -5y - 6x$

i.e. $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} y \\ -6x - 5y \end{bmatrix}$ which is 3.

Thus we also get the general sol'n to 3 by taking $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ x' \end{bmatrix}$ from 2a)

$$= \begin{bmatrix} c_1 e^{-3t} + c_2 e^{-2t} \\ -3c_1 e^{-3t} - 2c_2 e^{-2t} \end{bmatrix} = c_1 e^{-3t} \begin{bmatrix} 1 \\ -3 \end{bmatrix} + c_2 e^{-2t} \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

(by luck, my form is identical to 3a sol'n, yours could be equivalent but not identical)

4) Use Laplace transform to solve the initial value problem for the undamped resonator:

$$\frac{d^2}{dt^2}x(t) + \omega_0^2 x(t) = F_0 \cos(\omega_0 t)$$
$$x(0) = x_0$$
$$D(x)(0) = v_0$$

(10 points)

$$s^2 X(s) - s x_0 - v_0 + \omega_0^2 X(s) = F_0 \frac{s}{s^2+\omega_0^2}$$

$$X(s)\left[s^2+\omega_0^2\right] = F_0 \frac{s}{s^2+\omega_0^2} + s x_0 + v_0$$

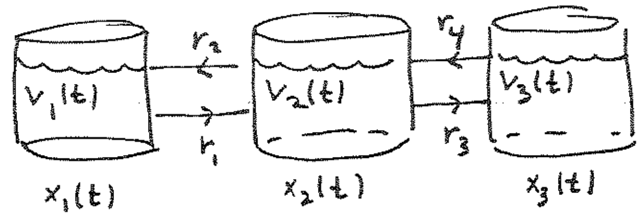
$$X(s) = F_0 \frac{s}{(s^2+\omega_0^2)^2} + x_0 \frac{s}{s^2+\omega_0^2} + {}^{v_0}\!/_{\omega_0} \frac{\omega_0}{s^2+\omega_0^2}$$

table!

$$\boxed{x(t) = F_0 \frac{t}{2\omega_0} \sin\omega_0 t + x_0 \cos\omega_0 t + \frac{v_0}{\omega_0} \sin\omega_0 t}$$

resonance term

5) Consider the following three-tank configuration. Let tank i have volume $V_i(t)$ and solute amount $x_i(t)$ at time t. Well-mixed liquid flows between tanks one and two, with rates r_1, r_2, and also between tanks two and three, with rates r_3, r_4, as indicated.



5a) What is the system of 6 first order differential equations governing the volumes $V_1(t)$, $V_2(t)$, $V_3(t)$ and solute amounts $x_1(t)$, $x_2(t)$, $x_3(t)$? (Hint: Although most of our recent tanks have had constant volume, how fast volume is changing depends on how fast volume is leaving and how fast it is coming in.)

(6 points)

$V_1' = r_2 - r_1$ vol/time

$V_2' = r_1 + r_4 - r_2 - r_3$

$V_3' = r_3 - r_4$

$x_1' = r_i c_i - r_o c_o = r_2 \frac{x_2}{V_2} - r_1 \frac{x_1}{V_1}$

$x_2' = r_1 \frac{x_1}{V_1} + r_4 \frac{x_3}{V_3} - r_2 \frac{x_2}{V_2} - r_3 \frac{x_2}{V_2}$

$x_3' = r_3 \frac{x_2}{V_2} - r_4 \frac{x_3}{V_3}$

5b) Suppose that all four rates are 100 gallons/hour, so that the volumes in each tank remain constant. Suppose that these volumes are each 100 gallons. Show that in this case, the differential equations in (5a) for the solute amounts reduce to the system

$V_1' = V_2' = V_3' = 0$, so V_i const

$$\begin{bmatrix} \frac{dx_1}{dt} \\ \frac{dx_2}{dt} \\ \frac{dx_3}{dt} \end{bmatrix} = \begin{bmatrix} -1 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

(4 points)

$V_1 = V_2 = V_3 = 100$

$r_1 = r_2 = r_3 = r_4 = 100$

(so each $\frac{r_i}{V_j} = 1$)

so

$x_1' = 100 \frac{x_2}{100} - 100 \frac{x_1}{100}$

$= -x_1 + x_2$

$x_2' = \frac{100}{100} x_1 + x_3 - x_2 - x_2 = x_1 - 2x_2 + x_3$

$x_3' = x_2 - x_3$

i.e. $\begin{bmatrix} x_1' \\ x_2' \\ x_3' \end{bmatrix} = \begin{bmatrix} -x_1 + x_2 \\ x_1 - 2x_2 + x_3 \\ x_2 - x_3 \end{bmatrix}$

matrix form

5c) Maple says that:

```
> A := matrix(3, 3, [-1, 1, 0, 1, -2, 1, 0, 1, -1]);
  eigenvectors(A);
```

$$A := \begin{bmatrix} -1 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -1 \end{bmatrix}$$

λ_i ($\lambda - \lambda_i$) ① eigenbasis

$$[0, 1, \{[1 \;\; 1 \;\; 1]\}], [-1, 1, \{[-1 \;\; 0 \;\; 1]\}], [-3, 1, \{[1 \;\; -2 \;\; 1]\}] \qquad (1)$$

Use this information to write the general solution to the system in (5b).

(5 points)

$$\vec{x}(t) = c_1 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + c_2 e^{-t} \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} + c_3 e^{-3t} \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

5d) Solve the initial value problem for the tank problem in (5b), assuming there are initially 10 pounds of solute in tank 1, 20 pounds in tank 2, and none in tank 3.

$x_1(0) = 10$ $\quad x_2(0) = 20 \quad x_3(0) = 0.$

(10 points)

$$c_1 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} + c_3 \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 10 \\ 20 \\ 0 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & 10 \\ 1 & 0 & -2 & 20 \\ 1 & 1 & 1 & 0 \end{array}\right]$$

$$\begin{array}{r} \\ -R_1+R_2 \\ -R_1+R_3 \end{array} \left[\begin{array}{ccc|c} 1 & -1 & 1 & 10 \\ 0 & 1 & -3 & 10 \\ 0 & 2 & 0 & -10 \end{array}\right]$$

$$\begin{array}{r} \\ R_2/2 \\ R_2 \end{array} \left[\begin{array}{ccc|c} 1 & -1 & 1 & 10 \\ 0 & 1 & 0 & -5 \\ 0 & 1 & -3 & 10 \end{array}\right]$$

$$\begin{array}{r} R_2+R_1 \\ \\ -R_2+R_3 \end{array} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 5 \\ 0 & 1 & 0 & -5 \\ 0 & 0 & -3 & 15 \end{array}\right]$$

$$\begin{array}{r} \\ \\ R_3/_{-3} \end{array} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 5 \\ 0 & 1 & 0 & -5 \\ 0 & 0 & 1 & -5 \end{array}\right]$$

$$\begin{array}{r} -R_3+R_1 \\ \\ \\ \end{array} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 10 \\ 0 & 1 & 0 & -5 \\ 0 & 0 & 1 & -5 \end{array}\right]$$

so

$$\boxed{\vec{x}(t) = 10 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} - 5e^{-t} \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} - 5e^{-3t} \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}}$$

(check: $\vec{x}(0) = \begin{bmatrix} 10+5-5 \\ 10+10 \\ 10-5-5 \end{bmatrix} = \begin{bmatrix} 10 \\ 20 \\ 0 \end{bmatrix}$ ✓

5e) What is the limiting amount of salt in each tank, as t approaches infinity? (Hint: You know this answer, no matter whether you actually solved 5d, but this gives a way of checking your work there.)

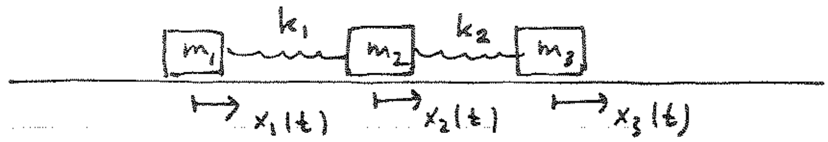
(5 points)

$$\lim_{t \to \infty} \vec{x}(t) = 10 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \quad \text{(exponentials} \to 0)$$

i.e. 10 lbs in each tank.

(Since there are 30 lbs total & 3 equal-volume tanks, we know we should end up with 10 lbs/tank as concentration becomes constant.)

6) Consider the following configuration of 3 masses held together with two springs, with positive displacements from equilibrium measured to the right, as usual. Notice that this train is not anchored to any wall.



6a) Use Newton's law and Hooke's usual linearization "law" to derive the system of 3 second order differential equations governing the masses' motion.

(5 points)

$$m_1 x_1'' = k_1(x_2 - x_1) \qquad = -k_1x_1 + k_1x_2$$
$$m_2 x_2'' = k_2(x_3 - x_2) - k_1(x_2 - x_1) \qquad = k_1x_1 - (k_1+k_2)x_2 + k_2x_3$$
$$m_3 x_3'' = -k_2(x_3 - x_2) \qquad = k_2x_2 - k_2x_3$$

6b) What is the dimension to the solution space to this problem? Explain.

(5 points)

6 dim'l : 3 2^{nd} order DE's convert to 6 1^{st} order DE's, & its a homogeneous syst. , so sol'n space is 6 d.

(alternately, the 6 initial values $x_1(0), x_1'(0), x_2(0), x_2'(0), x_3(0), x_3'(0)$ uniquely determine the solution.)

6c) Assume that all three masses are identical, and the two spring constants are also equal. Assume further that units have been chosen so that the numerical value "m" of each mass equals the numerical value "k" of each Hooke's constant. Show that in this case the system in (6a) reduces to

$$\begin{bmatrix} \frac{d^2}{dt^2}x_1(t) \\ \frac{d^2}{dt^2}x_2(t) \\ \frac{d^2}{dt^2}x_3(t) \end{bmatrix} = \begin{bmatrix} -1 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}.$$

(5 points)

from 6a), $\div m_i$:

$$x_1'' = -\frac{k_1}{m_1}x_1 + \frac{k_1}{m_1}x_2$$
$$x_2'' = \frac{k_1}{m_2}x_1 - \frac{(k_1+k_2)}{m_2}x_2 + \frac{k_2}{m_2}x_3$$
$$x_3'' = \frac{k_2}{m_3}x_2 - \frac{k_2}{m_3}x_3$$

matrix form

if each $\frac{k_i}{m_j} = 1$, this reduces to

$$x_1'' = -x_1 + x_2$$
$$x_2'' = x_1 - 2x_2 + x_3$$
$$x_3'' = x_2 - x_3$$

6d) Notice that the coefficient matrix in (6c) is the same one that appeared in problem (5). Use this information to write down the general solution to the system in (6c). As you recall,

```
> A := matrix(3, 3, [-1, 1, 0, 1, -2, 1, 0, 1, -1]);
  eigenvectors(A);
```

$\lambda = -1$
$\omega = 1$

$$A := \begin{bmatrix} -1 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -1 \end{bmatrix}$$

$\lambda = -3$
$\omega = \sqrt{3}$

$$[-1, 1, \{[\,-1 \;\; 0 \;\; 1\,]\}], [0, 1, \{[\,1 \;\; 1 \;\; 1\,]\}], [-3, 1, \{[\,1 \;\; -2 \;\; 1\,]\}]$$ (2)

$\vec{x}'' = A\vec{x}$

(6 points)

$\cos\omega t\,\vec{v}$, $\sin\omega t\,\vec{v}$ with $A\vec{v} = \lambda\vec{v}$, $\omega = \sqrt{-\lambda}$. Except when $\lambda = 0$, then $(c_1 + c_2 t)\vec{v}$ solves $\vec{x}'' = A\vec{x}$

$$\vec{x}(t) = (c_1 + c_2 t)\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + (c_3 \cos t + c_4 \sin t)\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} + (c_5 \cos\sqrt{3}t + c_6 \sin\sqrt{3}t)\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

6e) Describe the three "fundamental modes" for this mass-spring problem.

(6 points)

moving with constant speed c_2, with all springs at equilibrium displacement

mass 2 at rest, masses 1 & 3 oscillating out of phase with equal amplitudes

masses 1 & 3 in phase, but out of phase with mass 2, which has twice the amplitude.

6f) Now suppose there is an external sinusoidal force, so that the second order system becomes inhomogeneous,

$$\frac{d^2}{dt^2}\,x(t) = A\,x + \cos(\omega\,t)\;b\,.$$

Using matrix algebra (like we did in class, and you did in your Maple project), derive a formula for a particular solution

$$x_p(t) = \cos(\omega\,t)\;c$$

to this system. What do you expect will happen to this solution if ω is close to one of the natural frequencies?

$$\vec{x}'' = A\vec{x} + \cos\omega t\,\vec{b}$$

(8 points)

try $\vec{x}_p = \cos\omega t\,\vec{c}$

$\Rightarrow \vec{x}_p'' = -\omega^2 \cos\omega t\,\vec{c}$

$A\vec{x}_p + \cos\omega t\,\vec{b} = \cos\omega t\,A\vec{c} + \cos\omega t\,\vec{b}$

equate these, cancel $\cos\omega t$

$$\Leftrightarrow -\omega^2\vec{c} = A\vec{c} + \vec{b}$$

$$-\vec{b} = A\vec{c} + \omega^2 I\vec{c} = (A + \omega^2 I)\vec{c}$$

So $\boxed{\vec{c} = -(A + \omega^2 I)^{-1}\vec{b}}$ (whenever $A + \omega^2 I$ is invertible, i.e. whenever $\omega^2 \neq 0, 1, 3$).

For ω close to 1 or $\sqrt{3}$ or 0 we expect $\vec{c}(\omega)$ to get large entries (yields practical reson. in slightly damped problem)

7) Consider the system of differential equations below which models two populations x(t) and y(t):

$$\begin{bmatrix} \dfrac{dx}{dt} \\[2ex] \dfrac{dy}{dt} \end{bmatrix} = \begin{bmatrix} 9x - x^2 - 2xy \\ \underbrace{12y - y^2}_{\text{logistic}} - 2xy \end{bmatrix}$$

7a) If this was a model of two interacting populations, what kind would it be? Explain.

(2 points)

competition: x or y alone survive logistically.
But the neg. coeff's in front of xy in each eqtn indicate that the presence of either hurts the other.

7b) Find all four equilbrium solutions to this system of differential equations. (Hint: One of them is [5, 2].)

(8 points)

$x(9-x-2y)=0$

$y(12-y-2x)=0$

$x=0$

$\xrightarrow{y=0} \begin{bmatrix}0\\0\end{bmatrix}$

$\xrightarrow{12-y-2x=0} y=12 \quad \begin{bmatrix}0\\12\end{bmatrix}$

$9-x-2y=0$

$\xrightarrow{y=0} x=9 \quad \begin{bmatrix}9\\0\end{bmatrix}$

$\xrightarrow{12-y-2x=0}$

$x+2y=9$

$2x+y=12$

$$x = \frac{\begin{vmatrix} 9 & 2 \\ 12 & 1 \end{vmatrix}}{\begin{vmatrix} 1 & 2 \\ 2 & 1 \end{vmatrix}} = \frac{-15}{-3} = 5$$

$$y = \frac{\begin{vmatrix} 1 & 9 \\ 2 & 12 \end{vmatrix}}{-3} = \frac{-6}{-3} = 2$$

$\begin{bmatrix}5\\2\end{bmatrix}$

equil. sol'ns:

$$\begin{bmatrix}0\\0\end{bmatrix}, \begin{bmatrix}0\\12\end{bmatrix}, \begin{bmatrix}9\\0\end{bmatrix}, \begin{bmatrix}5\\2\end{bmatrix}$$

$\begin{array}{l} 34 \\ 170 \\ 25 \\ \hline 195 \end{array}$

7c) Find the linearization of the population model near the equilibrium solution [5,2]. Use eigenvalues for the linearization to classify the type of singularity in the nonlinear problem. For your convenience, the system is repeated below:

(10 points)

$$\begin{bmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \end{bmatrix} = \begin{bmatrix} 9x - x^2 - 2xy \\ 12y - y^2 - 2xy \end{bmatrix} \begin{array}{l} = F \\ = G \end{array}$$

$$J = \begin{bmatrix} F_x & F_y \\ G_x & G_y \end{bmatrix} = \begin{bmatrix} 9-2x-2y & -2x \\ -2y & 12-2y-2x \end{bmatrix}$$

$$J(5,2) = \begin{bmatrix} 9-10-4 & -10 \\ -4 & 12-4-10 \end{bmatrix} = \begin{bmatrix} -5 & -10 \\ -4 & -2 \end{bmatrix}$$

$$\boxed{\begin{bmatrix} u' \\ v' \end{bmatrix} = \begin{bmatrix} -5 & -10 \\ -4 & -2 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}}$$

$$|J - \lambda I| = \begin{vmatrix} -5-\lambda & -10 \\ -4 & -2-\lambda \end{vmatrix} = (\lambda+5)(\lambda+2) - 40 = \lambda^2 + 7\lambda - 30$$

$$= (\lambda+10)(\lambda-3)$$

$$\boxed{\lambda = -10, 3}$$

$\lambda_1 < 0 < \lambda_2$ (unstable)

Saddle

7d) Finding and using the eigenvectors from the linearization above, write the general solution $\begin{bmatrix} u(t) \\ v(t) \end{bmatrix}$ of the linearized problem at [5,2].

(10 points)

$\lambda = -10$

$$\left[\begin{array}{cc|c} 5 & -10 & 0 \\ -4 & 8 & 0 \\ \hline 1 & -2 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$$\vec{v} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$\lambda = 3$

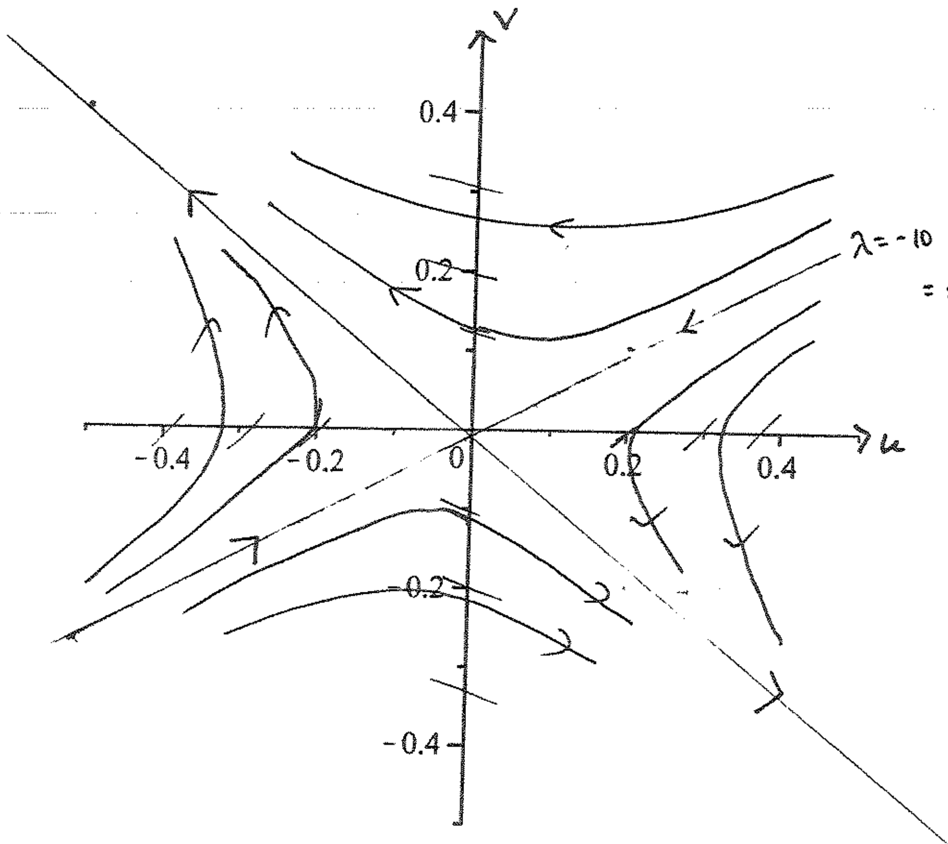
$$\left[\begin{array}{cc|c} -8 & -10 & 0 \\ -4 & -5 & 0 \\ \hline 4 & 5 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$$\vec{v} = \begin{bmatrix} -5 \\ 4 \end{bmatrix}$$

$$\begin{bmatrix} u \\ v \end{bmatrix} = c_1 e^{-10t} \begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2 e^{3t} \begin{bmatrix} -5 \\ 4 \end{bmatrix}$$

7e) Sketch the phase portrait for the linearized problem, using your work in (7d). Hint: the eigendirections should appear in your sketch.

(5 points)



$\lambda = -10$ eigenspace $= \text{span}\left\{\begin{bmatrix}2\\1\end{bmatrix}\right\}$

$\lambda = 3$ eigenspace $= \text{span}\left\{\begin{bmatrix}-5\\4\end{bmatrix}\right\}$

$$\begin{bmatrix}u\\v\end{bmatrix} = c_1 e^{-10t}\begin{bmatrix}2\\1\end{bmatrix} + c_2 e^{3t}\begin{bmatrix}-5\\4\end{bmatrix}$$

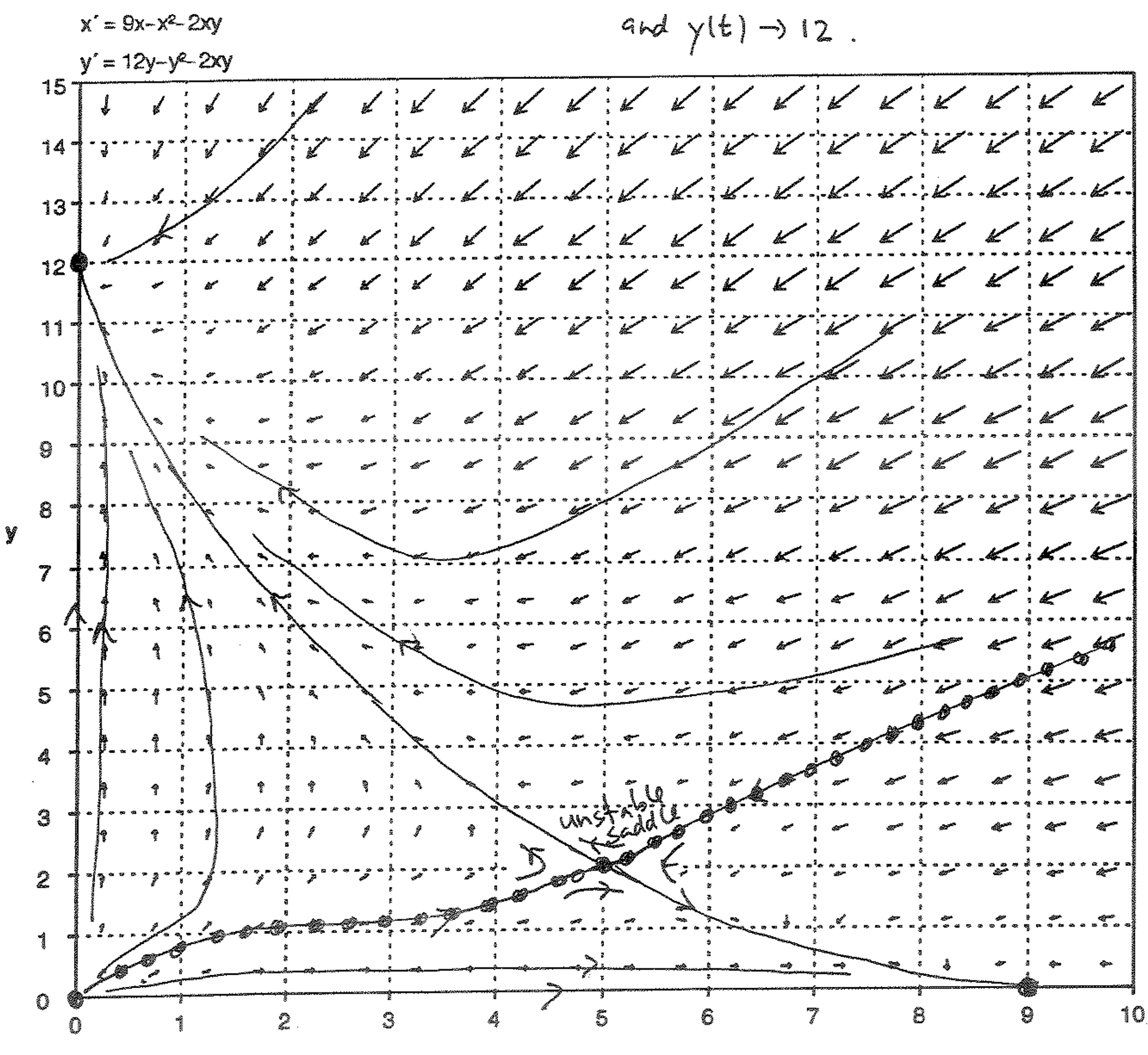
$$\begin{bmatrix}u'\\v'\end{bmatrix} = \begin{bmatrix}-5 & -10\\-4 & -2\end{bmatrix}\begin{bmatrix}u\\v\end{bmatrix}$$

7f) Use your work from (7e) to fill the portion of the phase portrait below which has been excised. Then describe what happens to solutions to the initial value problem for this system of DE's, depending on initial populations. You may wish to sketch representative solution curves, and indicate regions of the first quadrant in order to help you answer this part.

(5 points)

too much competition!

if init pops above dotted line
$x(t)$ dies out
and $y(t) \to 12$.



if init pops below dotted curve, $y(t)$ dies out
& $x(t) \to 9$