Name..

I.D. number..

Math 2250-1
FINAL EXAM
Dec 16, 2008

This exam is closed-book and closed-note. You may use a scientific calculator, but not one which is capable of graphing or of solving differential or linear algebra equations. Laplace Transform Tables are included with this exam. **In order to receive full or partial credit on any problem, you must show all of your work** and **justify your conclusions.** This exam counts for 30% of your course grade. It has been written so that there are 200 points possible, however, and the point values for each problem are indicated in the right-hand margin. **Good Luck!**

problem	score	possible
1	_______	35
2	_______	30
3	_______	20
4	_______	10
5	_______	30
6	_______	35
7	_______	40
Total	_______	200

1a) A motorboat containing the pilot has total mass of 360 kilograms, and its motor is able to provide 120 Newtons of thrust. However, when the boat is in motion, drag from the water produces a force of 6 newtons for each meter/sec of boat velocity. Use Newton's law to explain why (while the motor is on) the boat velocity satisfies the differential equation

$$\frac{dv}{dt} = \frac{1}{3} - \frac{v}{60}$$

(5 points)

1b) What is the equilibrium solution for the velocity v(t)? Is this solution stable or unstable? Explain with a phase diagram.

(5 points)

1c) Solve the initial value problem for the boat's velocity, assuming the boat starts at rest. Use the integrating-factor method we learned in Chapter 1,for first order linear differential equations.

(10 points)

1d) Resolve the same IVP, this time using the algorithm for separable differential equations.

(10 points)

1d) If the boat starts at rest, how long does it take to reach 75% of its terminal velocity?

(5 points)

2) Consider the following differential equation, which could arise as an unforced forced spring problem:

$$x''(t) + 5\,x'(t) + 6\,x(t) = 0.$$

2a) Use the characteristic polynomial to find the general solution to this differential equation. What kind of damping governs this DE?

(8 points)

2b) Use your work in (2a) to solve the initial value problem

$$x''(t) + 5\,x'(t) + 6\,x(t) = 0$$
$$x(0) = 1$$
$$x'(0) = -1$$

(12 points)

2c) Resolve the initial value problem above, using Laplace transforms.

(10 points)

3) Consider the first order system of differential equations

$$\begin{bmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -6 & -5 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}.$$

3a) Find the general solution to this system, using the eigenvalue/eigenvector method.

(15 points)

3b) Explain how this system is related to the second order differential equation in problem (2), and how you could have deduced the general solution in (3a) directly from the one you found in (2b).

(5 points)

4) Use Laplace transform to solve the initial value problem for the undamped resonator:

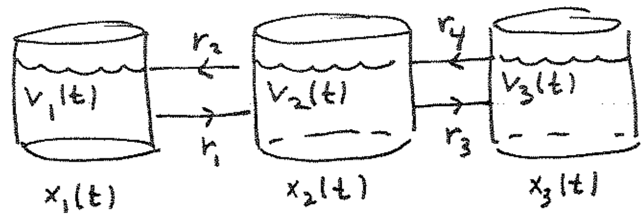
$$\frac{d^2}{dt^2}x(t) + \omega_0^2 x(t) = F_0 \cos(\omega_0 t)$$

$$x(0) = x_0$$

$$D(x)(0) = v_0$$

(10 points)

5) Consider the following three-tank configuration. Let tank i have volume $V_i(t)$ and solute amount $x_i(t)$ at time t. Well-mixed liquid flows between tanks one and two, with rates r_1, r_2, and also between tanks two and three, with rates r_3, r_4, as indicated.



5a) What is the system of 6 first order differential equations governing the volumes $V_1(t)$, $V_2(t)$, $V_3(t)$ and solute amounts $x_1(t)$, $x_2(t)$, $x_3(t)$? (Hint: Although most of our recent tanks have had constant volume, how fast volume is changing depends on how fast volume is leaving and how fast it is coming in.)

(6 points)

5b) Suppose that all four rates are 100 gallons/hour, so that the volumes in each tank remain constant. Suppose that these volumes are each 100 gallons. Show that in this case, the differential equations in (5a) for the solute amounts reduce to the system

$$\begin{bmatrix} \frac{dx_1}{dt} \\ \frac{dx_2}{dt} \\ \frac{dx_3}{dt} \end{bmatrix} = \begin{bmatrix} -1 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

(4 points)

5c) Maple says that:

```
> A := matrix(3, 3, [-1, 1, 0, 1,-2, 1, 0, 1,-1]);
  eigenvectors(A);
```

$$A := \begin{bmatrix} -1 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -1 \end{bmatrix}$$

$$[0, 1, \{[\ 1\ \ 1\ \ 1\ ]\}], [-1, 1, \{[\ -1\ \ 0\ \ 1\ ]\}], [-3, 1, \{[\ 1\ \ -2\ \ 1\ ]\}] \qquad (1)$$

Use this information to write the general solution to the system in (5b).

(5 points)

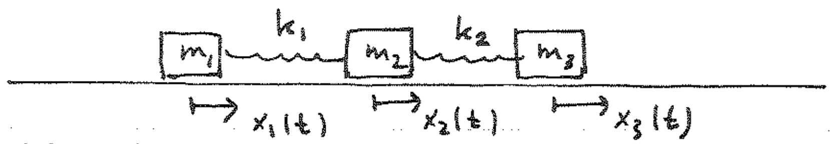
5d) Solve the initial value problem for the tank problem in (5b), assuming there are initially 10 pounds of solute in tank 1, 20 pounds in tank 2, and none in tank 3.

(10 points)

5e) What is the limiting amount of salt in each tank, as t approaches infinity? (Hint: You know this answer, no matter whether you actually solved 5d, but this gives a way of checking your work there.)

(5 points)

6) Consider the following configuration of 3 masses held together with two springs, with positive displacements from equilibrium measured to the right, as usual. Notice that this train is not anchored to any wall.



6a) Use Newton's law and Hooke's usual linearization "law" to derive the system of 3 second order differential equations governing the masses' motion.

(5 points)

6b) What is the dimension to the solution space to this problem? Explain.

(5 points)

6c) Assume that all three masses are identical, and the two spring constants are also equal. Assume further that units have been chosen so that the numerical value "m" of each mass equals the numerical value "k" of each Hooke's constant. Show that in this case the system in (6a) reduces to

$$\begin{bmatrix} \frac{d^2}{dt^2} x_1(t) \\ \frac{d^2}{dt^2} x_2(t) \\ \frac{d^2}{dt^2} x_3(t) \end{bmatrix} = \begin{bmatrix} -1 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}.$$

(5 points)

6d) Notice that the coefficient matrix in (6c) is the same one that appeared in problem (5). Use this information to write down the general solution to the system in (6c). As you recall,

```
> A := matrix(3, 3, [-1, 1, 0, 1,-2, 1, 0, 1,-1]);
  eigenvectors(A);
```

$$A := \begin{bmatrix} -1 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -1 \end{bmatrix}$$

$$[-1, 1, \{[\ -1\ \ 0\ \ 1\ ]\}], [0, 1, \{[\ 1\ \ 1\ \ 1\ ]\}], [-3, 1, \{[\ 1\ \ -2\ \ 1\ ]\}] \qquad \textbf{(2)}$$

(6 points)

6e) Describe the three "fundamental modes" for this mass-spring problem.

(6 points)

6f) Now suppose there is an external sinusoidal force, so that the second order system becomes inhomogeneous,

$$\frac{d^2}{dt^2}x(t) = A\,x + \cos(\omega\, t)\, b\,.$$

Using matrix algebra (like we did in class, and you did in your Maple project), derive a formula for a particular solution

$$x_p(t) = \cos(\omega\, t)\, c$$

to this system. What do you expect will happen to this solution if ω is close to one of the natural frequencies?

(8 points)

7) Consider the system of differential equations below which models two populations x(t) and y(t):

$$\begin{bmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \end{bmatrix} = \begin{bmatrix} 9x - x^2 - 2xy \\ 12y - y^2 - 2xy \end{bmatrix}$$

7a) If this was a model of two interacting populations, what kind would it be? Explain.

(2 points)

7b) Find all four equilbrium solutions to this system of differential equations. (Hint: One of them is [5, 2].)

(8 points)

7c) Find the linearization of the population model near the equilibrium solution [5,2]. Use eigenvalues for the linearization to classify the type of singularity in the nonlinear problem. For your convenience, the system is repeated below:

(10 points)

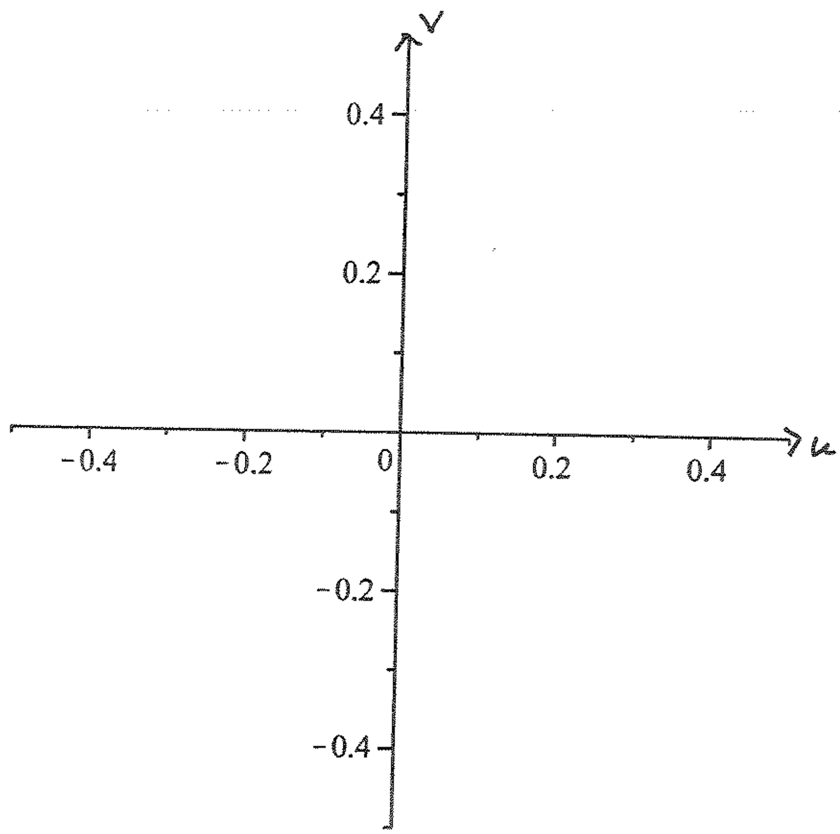
$$\begin{bmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \end{bmatrix} = \begin{bmatrix} 9x - x^2 - 2xy \\ 12y - y^2 - 2xy \end{bmatrix}$$

7d) Finding and using the eigenvectors from the linearization above, write the general solution $\begin{bmatrix} u(t) \\ v(t) \end{bmatrix}$ of the linearized problem at [5,2].

(10 points)

7e) Sketch the phase portrait for the linearized problem, using your work in (7d). Hint: the eigendirections should appear in your sketch.

(5 points)



7f) Use your work from (7e) to fill the portion of the phase portrait below which has been excised. Then describe what happens to solutions to the initial value problem for this system of DE's, depending on initial populations. You may wish to sketch representative solution curves, and indicate regions of the first quadrant in order to help you answer this part.

(5 points)

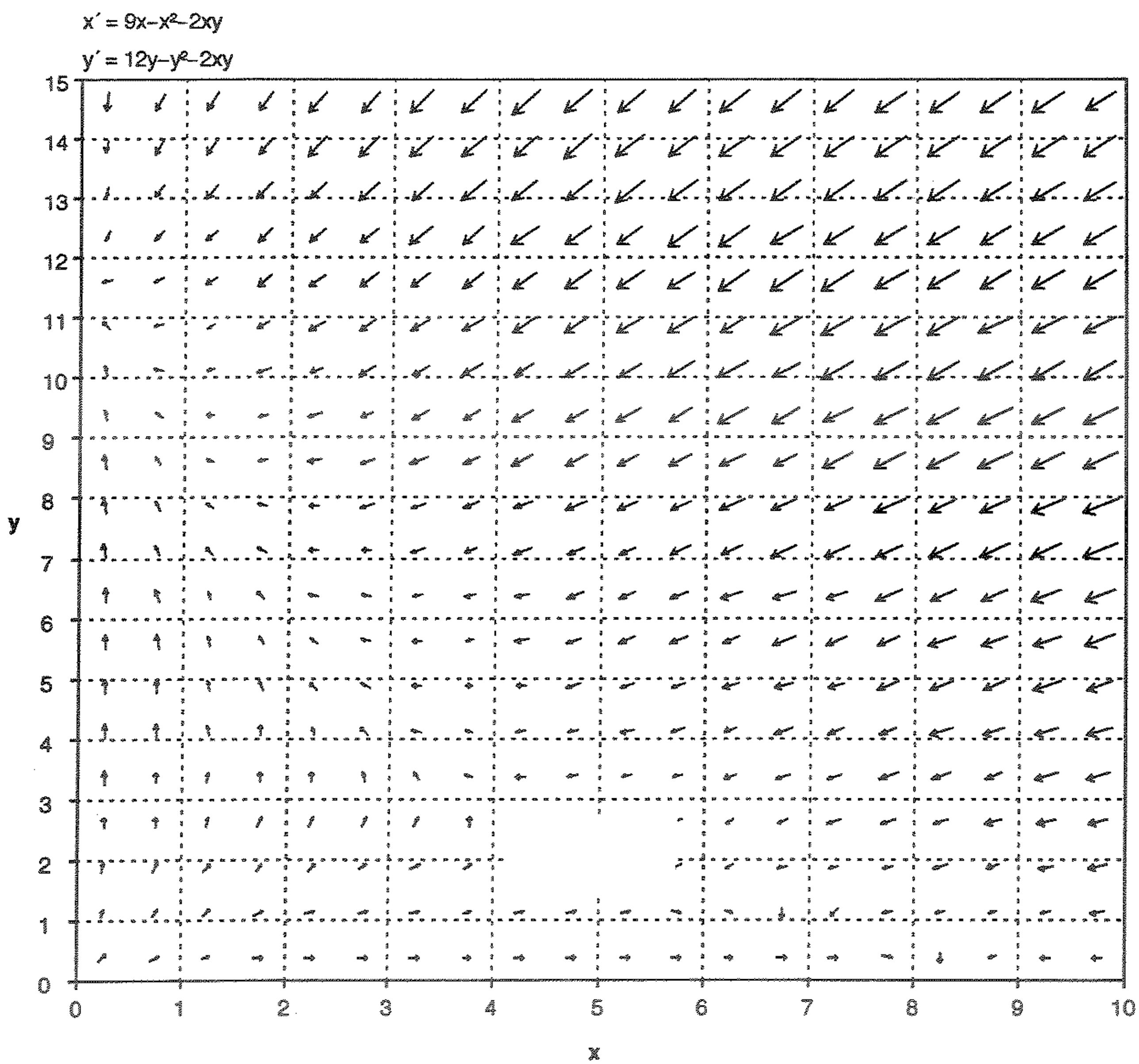


Table of Laplace Transforms

This table summarizes the general properties of Laplace transforms and the Laplace transforms of particular functions derived in Chapter 10.

Function	Transform	Function	Transform
$f(t)$	$F(s)$	e^{at}	$\dfrac{1}{s-a}$
$af(t)+bg(t)$	$aF(s)+bG(s)$	$t^n e^{at}$	$\dfrac{n!}{(s-a)^{n+1}}$
$f'(t)$	$sF(s)-f(0)$	$\cos kt$	$\dfrac{s}{s^2+k^2}$
$f''(t)$	$s^2F(s)-sf(0)-f'(0)$	$\sin kt$	$\dfrac{k}{s^2+k^2}$
$f^{(n)}(t)$	$s^nF(s)-s^{n-1}f(0)-\cdots-f^{(n-1)}(0)$	$\cosh kt$	$\dfrac{s}{s^2-k^2}$
$\int_0^t f(\tau)\,d\tau$	$\dfrac{F(s)}{s}$	$\sinh kt$	$\dfrac{k}{s^2-k^2}$
$e^{at}f(t)$	$F(s-a)$	$e^{at}\cos kt$	$\dfrac{s-a}{(s-a)^2+k^2}$
$u(t-a)f(t-a)$	$e^{-as}F(s)$	$e^{at}\sin kt$	$\dfrac{k}{(s-a)^2+k^2}$
$\int_0^t f(\tau)g(t-\tau)\,d\tau$	$F(s)G(s)$	$\dfrac{1}{2k^3}(\sin kt - kt\cos kt)$	$\dfrac{1}{(s^2+k^2)^2}$
$tf(t)$	$-F'(s)$	$\dfrac{t}{2k}\sin kt$	$\dfrac{s}{(s^2+k^2)^2}$
$t^n f(t)$	$(-1)^n F^{(n)}(s)$	$\dfrac{1}{2k}(\sin kt + kt\cos kt)$	$\dfrac{s^2}{(s^2+k^2)^2}$
$\dfrac{f(t)}{t}$	$\int_s^\infty F(\sigma)\,d\sigma$	$u(t-a)$	$\dfrac{e^{-as}}{s}$
$f(t)$, period p	$\dfrac{1}{1-e^{-ps}}\int_0^p e^{-st}f(t)\,dt$	$\delta(t-a)$	e^{-as}
1	$\dfrac{1}{s}$	$(-1)^{[\![t/a]\!]}$ (square wave)	$\dfrac{1}{s}\tanh\dfrac{as}{2}$
t	$\dfrac{1}{s^2}$	$\left[\!\left[\dfrac{t}{a}\right]\!\right]$ (staircase)	$\dfrac{e^{-as}}{s(1-e^{-as})}$
t^n	$\dfrac{n!}{s^{n+1}}$		
$\dfrac{1}{\sqrt{\pi t}}$	$\dfrac{1}{\sqrt{s}}$		
t^a	$\dfrac{\Gamma(a+1)}{s^{a+1}}$		