Name ... SOLUTIONS ...

Student I.D. number ...

# Math 2250-1

Exam 2

November 17, 2008

The exam is closed book and closed note, but a scientific calculator is allowed and may be useful. Calculators which can do linear algebra or solve differential equations are not allowed. In order to receive partial credit or full credit all work must be shown. There are 100 points possible on the test, and the point values of each problem are indicated in the right margin. You may wish to ration your time accordingly. GOOD LUCK!

| Score: | Possible |
|---|---|
| 1________ | 20 |
| 2________ | 30 |
| 3________ | 40 |
| 4________ | 10 |
| Total________ | 100 |

1) Here is a matrix A (on the left), and on the right is its reduced row echelon form:

$$A := \begin{bmatrix} 1 & 3 & 0 & 1 & 1 \\ -1 & -3 & 0 & -1 & -1 \\ 2 & 6 & 1 & 3 & 0 \\ -2 & -6 & 2 & 0 & -6 \\ 3 & 9 & 1 & 4 & 1 \end{bmatrix} \qquad \begin{array}{cc} & \begin{array}{ccccc} & r & & s & t \\ x_1 & x_2 & x_3 & x_4 & x_5 \end{array} \\ reduced := & \begin{bmatrix} 1 & 3 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{array}{c} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{array} \end{array}$$

1a) Find all solutions to the homogeneous linear equation Ax=0. (10 points)

$x_1 = -3r - s - t$

$x_2 = r$

$x_3 = -s + 2t$

$x_4 = s$

$x_5 = t$

backsolve

$$\vec{x} = r\begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + s\begin{bmatrix} -1 \\ 0 \\ -1 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} -1 \\ 0 \\ 2 \\ 0 \\ 1 \end{bmatrix}$$

1b) Exhibit a basis for the solution space you found in part 1a). (5 points)

$$\left\{ \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 2 \\ 0 \\ 1 \end{bmatrix} \right\}$$

1c) Prove that the vectors you exhibited in (1b) are indeed a basis for the solution space. Your explanation should make it clear that you know what it means for a collection of vectors to be a basis! (5 points)

- span: every soltn to $A\vec{x} = \vec{0}$ is a linear combo of the 3 vectors in the basis, this is shown in (a)

- linear independence:

$$\text{if} \quad c_1\begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + c_2\begin{bmatrix} -1 \\ 0 \\ -1 \\ 1 \\ 0 \end{bmatrix} + c_3\begin{bmatrix} -1 \\ 0 \\ 2 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \begin{array}{c} \\ \leftarrow \\ \\ \leftarrow \\ \leftarrow \end{array}$$

then eqtn 2: $c_1 = 0$

4: $c_2 = 0$

5 $c_3 = 0$

so $c_1 = c_2 = c_3 = 0$

2) Consider the differential equation

$$y'''(x) + 3y''(x) - 4y(x) = 0$$

2a) Find the general solution to this differential equation. (Hint: $y(x) = e^x$ is one solution to this DE.)

(15 points)

$y(x) = e^{rx}$ : $r^3 + 3r^2 - 4 = 0$

$r = 1$ is a root

$(1 + 3 - 4 = 0 \checkmark)$

$$\begin{array}{r} r^2 + 4r + 4 \\ r-1 \enclose{longdiv}{r^3 + 3r^2 \qquad - 4} \\ r^3 - r^2 \qquad\qquad \\ \hline 4r^2 \qquad - 4 \\ 4r^2 - 4r \qquad \\ \hline 4r - 4 \\ 4r - 4 \\ \hline 0 \end{array}$$

$= (r+2)^2$

so $p(r) = (r-1)(r+2)^2$

so $\boxed{y_H(x) = c_1 e^x + c_2 e^{-2x} + c_3 x e^{-2x}}$

2b) Suppose you wanted to solve the IVP

$$y'''(x) + 3y''(x) - 4y(x) = 0$$
$$y(0) = b_1$$
$$y'(0) = b_2$$
$$y''(0) = b_3$$

Find the system of equations you would need to solve for the linear combination coefficients (which you probably called $c_1, c_2, c_3$) in (2a), in order to solve the IVP. Do a computation to verify that this matrix system has a unique solution - (but you don't need to actually find the solution) ! As always, explain your reasoning.

(5 points)

$y = c_1 e^x + c_2 e^{-2x} + c_3 x e^{-2x}$

$y' = c_1 e^x - 2c_2 e^{-2x} + c_3 [e^{-2x} - 2x e^{-2x}]$

$y'' = c_1 e^x + 4c_2 e^{-2x} + c_3 [-4e^{-2x} + 4x e^{-2x}]$

so we need to solve (@ $x = 0$):

$$\begin{aligned} c_1 + c_2 \qquad &= b_1 \\ c_1 - 2c_2 + c_3 &= b_2 \\ c_1 + 4c_2 - 4c_3 &= b_3 \end{aligned}$$

$$\underbrace{\begin{bmatrix} 1 & 1 & 0 \\ 1 & -2 & 1 \\ 1 & 4 & -4 \end{bmatrix}}_{[W(0)]} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

the coefficient matrix is the Wronskian matrix at $x = 0$. If its det $\neq 0$ then it has an inverse, and $\vec{c} = [\ ]^{-1} \vec{b}$

at $x = 0$,

$|W| = 8 + 1 - 4 + 4 = 9 \neq 0$

so unique $\vec{c}$ exists

2c) What is the dimension of the solution space to the homogeneous differential equation in this problem? As part of your explanation, exhibit a basis for the solution space.

( 5 points)

3-d.

basis $\{e^{x}, e^{-2x}, xe^{-2x}\}$

2d) Assuming any general theorem we discussed in class or used in homework, use your work in (2b) to explain why the functions you listed in (2c) are a basis for the solution space.

(5 points)

- we know soltn space is 3-d, so if the 3 fcns above are independent they are a basis. The Wronskian det is $2b \neq 0$ (at $x=0$, actually everywhere) so they are. (3 independent vectors in a 3-d space automatically span

Or ~~know de~~

- if $c_1y_1 + c_2y_2 + c_3y_3 = 0$ then $c_1y_1' + c_2y_2' + c_3y_3' = 0$ $c_1y_1'' + c_2y_2'' + c_3y_3'' = 0$

at $x=0$ this gives $\begin{bmatrix} 1 & 1 & 0 \\ 1 & -2 & 1 \\ 1 & 4 & -4 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$\Rightarrow \vec{c} = \vec{0}$ since $\det \neq 0$. (as in 1st explanation) shows independence

Same combutation in 2b shows we can solve every IVP at $x=0$, with linear combos of $y_1, y_2, y_3$. Since soltns to IVP's are unique, and since every soltn solves some IVP, every soltn is a linear combo of $y_1, y_2, y_3$ shows span

3) Consider this differential equation, which could arise as a mass-spring (or electrical circuit) problem.

$$x''(t) + 4x(t) = 0$$

3a) If this was a mass-spring model, and if the mass was 4 kg, identify the values for the damping coefficient and the Hooke's constant. Include correct units.

(4 points)

$$4x'' + 16x = 0$$

$c = 0$ kg/sec

$k = 16$ N/m

3b) What is the general solution to this homogeneous differential equation?

(6 points)

$$x(t) = A\cos 2t + B\sin 2t = C\cos(2t-\alpha)$$

since $x'' + \omega_0^2 x(t)$ has sol'n $A\cos\omega_0 t + B\sin\omega_0 t$

– we got this from charact poly,

since $r^2 + \omega_0^2 = 0 \Rightarrow r^2 = -\omega_0^2$

$r = \pm i\omega_0$

$$e^{i\omega_0 t} = \underbrace{\cos\omega_0 t}_{\text{one sol'n}} + i\underbrace{\sin\omega_0 t}_{\text{an ind sol'n}}$$

3c) Now consider the initial value problem:

$$x''(t) + 4x(t) = 16\sin(2t).$$
$$x(0) = -1$$
$$x'(0) = 2$$

Find the solution to this IVP, using Laplace transform techniques.

(15 points)

$$s^2 X(s) - s(-1) - 2 + 4X(s) = 16 \cdot \frac{2}{s^2+4}$$

$$X(s)(s^2+4) = \frac{32}{s^2+4} + -s+2$$

$$X(s) = \frac{32}{(s^2+4)^2} - \frac{s}{s^2+4} + \frac{2}{s^2+4}$$

from table,

$$x(t) = 32 \cdot \frac{1}{16}(\sin 2t - 2t\cos 2t) - \cos 2t + \sin 2t$$

$$\boxed{x(t) = -4t\cos 2t + 3\sin 2t - \cos 2t}$$

3d) What physical phenomenon is illustrated by the solution to (3c) ? In terms of the general mass-spring system, in what cases does this phenomenon happen?

(5 points)

resonance

~~occ~~ occurs when $c = 0$ (no damping)

and $\omega = \omega_0$ (driving frequence = natural frequency)

3e) Resolve the IVP in 3c), using the Chapter 5 techniques based on particular solutions and homogeneous solutions. You may use your work from 3b), and you may even steal your particular solution from your work in (3c), as long as you write down and justify the form your particular solution guess would take if you were using the method of undetermined coefficients. For your convenience, here is the IVP copied from the previous page:

$$x''(t) + 4x(t) = 16\sin(2t).$$
$$x(0) = -1$$
$$x'(0) = 2$$

(10 points)

try $x_p(t) = t\,(A\cos 2t + B\sin 2t)$

because solves homog eqtn.

ans from 3c) is

$x_p(t) = -4t\cos 2t$

so

$x(t) = -4t\cos 2t + A\cos 2t + B\sin 2t$

$x(0) = -1 = A$ so $A = -1$

$x'(0) = 2 = -4 + 2B$

$2B = 6$

$B = 3$

$$\boxed{x(t) = -4t\cos 2t - \cos 2t + 3\sin 2t}$$

4) Find the function x(t) which has Laplace transform

$$X(s)=\frac{5}{s^3+2s^2+5s} = \frac{5}{s(s^2+2s+5)} = \frac{A}{s}+\frac{Bs+C}{s^2+2s+5}$$

(10 points)

you could also do this as a convolution (but it's messy)

$$5 = A(s^2+2s+5)+(Bs+C)s$$

$$s=0 \Rightarrow 5=5A \quad \text{so } A=1$$

$$\begin{aligned} 5 &= s^2+2s+5+(Bs+C)s \\ &= s^2[1+B] \\ &\quad + s[2+C] \\ &\quad + 1[5] \end{aligned}$$

$$\begin{aligned} 1+B&=0 \\ 2+C&=0 \end{aligned} \Rightarrow B=-1, \quad C=-2$$

$$\begin{aligned} X(s) &= \frac{1}{s}+\frac{-s-2}{(s+1)^2+4} \\ &= \frac{1}{s}+\frac{-(s+1)-1}{(s+1)^2+4} \\ &= \frac{1}{s} - \underbrace{\frac{s+1}{(s+1)^2+4}}_{\mathcal{L}\{\cos 2t\}(s+1)} - \frac{1}{2}\underbrace{\frac{2}{(s+1)^2+4}}_{\mathcal{L}\{\sin 2t\}(s-1)} \end{aligned}$$

$$\mathcal{L}^{-1}: \quad e^{-t}\cos 2t \qquad e^{-t}\sin 2t$$

$$\boxed{x(t) = 1 - e^{-t}\cos 2t - \frac{1}{2}e^{-t}\sin 2t}$$