Name..
I.D. number...

Solutions

Math 2250-1
Exam 1
October 6, 2008

This exam is closed-book and closed-note. You may use a scientific calculator, but not one which is capable of graphing or of solving differential or linear algebra equations. **In order to receive full or partial credit on any problem, you must show all of your work** and **justify your conclusions.** There are 100 points possible. The point values for each problem are indicated in the right-hand margin.
Good Luck!

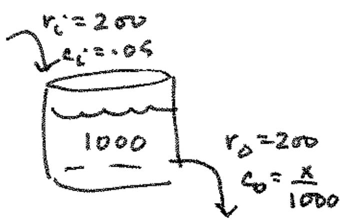
Score:	Possible
1__________	35
2__________	10
3__________	20
4__________	10
5__________	25
Total__________	100

1) Consider a brine tank which holds 1000 gallons of continuously-mixed liquid. Let x(t) be the amount of salt (in pounds) in the tank at time t. The in-flow and out-flow rates are both 200 gallons/hour, and the concentration of salt flowing into the tank is 0.05 pounds per gallon.

1a) Use the information above and your modeling ability to derive the differential equation for x(t):

$$\frac{dx}{dt} = 10 - 0.2\,x$$

(5 points)



$$\frac{dx}{dt} = r_i c_i - r_o c_o$$

$$\frac{dx}{dt} = 200(.05) - 200\,\frac{x}{1000}$$

$$\boxed{\frac{dx}{dt} = 10 - .2x}$$

1b) What is the equilibrium solution to this differential equation? Is it stable or unstable? Explain using a phase portrait.

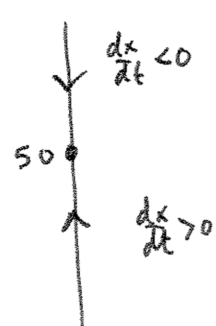
(5 points)

$10 - .2x = 0$

$.2x = 10$

$x = 50$ gal.

$$10 - .2x = -.2\left(x - \frac{10}{.2}\right)$$
$$= -.2(x - 50)$$



all sol'ns converge to 50

$\boxed{x = 50 \text{ is stable}}$

1c) Find the solution to the initial value problem for this differential equation,

$$\frac{dx}{dt} = 10 - 0.2\,x\,,$$

assuming there are originally 30 pounds of salt in the tank. This is a linear differential equation, so use the algorithm you learned for this sort of DE.

(10 points)

$x_0 = 30$

$$\frac{dx}{dt} + .2x = 10$$

$$e^{.2t}(\qquad) = 10\,e^{.2t}$$

$$(e^{.2t}x)' = 10e^{.2t}$$

$$e^{.2t}x = \int 10e^{.2t}\,dt = \frac{10}{.2}e^{.2t} + C$$

$$= 50e^{.2t} + C$$

$\div e^{.2t}$: $\quad x = 50 + Ce^{-.2t}$

$$x_0 = 30 = 50 + C \Rightarrow C = -20$$

$$\boxed{x(t) = 50 - 20e^{-.2t}}$$

1d) The differential equation we are considering in this problem is also separable. Use the algorithm you learned for separable DEs to resolve the initial value problem in (1c).

(10 points)

$$\frac{dx}{dt} = -.2(x-50)$$

$$\frac{dx}{x-50} = -.2\,dt$$

int: $\quad \ln|x-50| = -.2t + C_1$

$$|x-50| = e^{C_1}e^{-.2t}$$

$$x - 50 = Ce^{-.2t}$$

$$x = 50 + Ce^{-.2t}$$

$x_0 = 30$ so $C = -20$

$$\boxed{x = 50 - 20\,e^{-.2t}}$$

1e) For the solution you find in (1cd), is $\lim_{t\to\infty} x(t)$ consistent with your discussion in (1b)? How could your smart little sister have figured out the limiting amount of salt in the tank without understanding any differential equations, i.e. just by using the in-flow concentration?

(5 points)

yes; $\lim_{t\to\infty} x(t) = 50$, since $e^{-.2t} \to 0$ as $t \to \infty$.

limiting concentration = input concent = .05 lb/gal

so $\quad x(t) \to [(.05)\text{ lb/gal}][1000\text{ gal}]$

$= 50$ lb.

2a) Your father has a "sporty" car of mass 1000 kg and the engine and transmission of this car is able to provide a maximum force of 10,000 Newtons ($\frac{kg\ m}{sec^2}$) at all speeds. Assume the frictional forces on the car from air and road resistance are proportional to the car's velocity, i.e. we are studying a linear drag model. What is the drag coefficient, and what are its units, if the velocity of the car satisfies

$$\frac{dv}{dt} = 10 - 0.2\, v \ ?$$

(5 points)

$$m\frac{dv}{dt} = 10{,}000 - kv$$

$$1000\frac{dv}{dt} = 10{,}000 - kv$$

$$\frac{dv}{dt} = 10 - \frac{k}{1000}v$$

$$= 10 - .2v$$

so $\frac{k}{1000} = .2$

$$\boxed{k = 200 \quad N/m/sec.}$$

2b) Notice that you've ended up with the same differential equation as in problem 1 on this test! Suppose you're driving your Dad's car at a raceway, going 30 m/sec, and you begin to accelerate at the maximum rate possible. How long will it until you are going 45 m/sec (about 101 miles per hour)? Use your solution from 1d) or 1e)!

(5 points)

from 1(cd),

$$v(t) = 50 - 20e^{-.2t}$$

$$v(t) = 45 = 50 - 20e^{-.2t}$$

$$20e^{-.2t} = 5$$

$$e^{-.2t} = \frac{1}{4}$$

$$-.2t = \ln\left(\tfrac{1}{4}\right) = -\ln 4$$

$$t = \frac{\ln 4}{.2} \approx 6.93 \text{ sec}$$

3) Consider the following matrix equation, of the form Ax=b:

$$\begin{bmatrix} 2 & -1 & 2 \\ 1 & 1 & 1 \\ 1 & -5 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 3 \\ 3 \end{bmatrix}$$

3a) Compute the determinant of the coefficient matrix A above. What this tells you about the matrix equation and its possible solutions. In particular, can you rule out any of the three general possibilities: one solution, no solutions, or infinitiely many solutions, based on your determinant computation? Explain.

(8 points)

top row: $|A| = 2\begin{vmatrix} 1 & 1 \\ -5 & 1 \end{vmatrix} - -1\begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} + 2\begin{vmatrix} 1 & 1 \\ 1 & -5 \end{vmatrix}$

$= 2(6) + 0 + 2(-6)$

$= 0$

$|A| = 0$ means sol'ns may not exist, and if they do there are ∞'ly many

so, we can rule out: one solution

3b) Compute the reduced row echelon form of the 3 by 4 augmented matrix associated to the matrix equation above, and use it to find all solutions to the system.

(12 points)

$$\left[\begin{array}{ccc|c} 2 & -1 & 2 & 6 \\ 1 & 1 & 1 & 3 \\ 1 & -5 & 1 & 3 \end{array}\right]$$

$$\begin{array}{c} R_2 \\ R_1 \\ \end{array}\left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 2 & -1 & 2 & 6 \\ 1 & -5 & 1 & 3 \end{array}\right]$$

$$\begin{array}{c} \\ -2R_1+R_2 \\ -R_1+R_3 \end{array}\left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & -3 & 0 & 0 \\ 0 & -6 & 0 & 0 \end{array}\right]$$

$$\begin{array}{c} \\ R_2/-3 \\ \end{array}\left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\begin{array}{c} 6R_2+R_3 \\ -R_2+R_1 \\ \end{array}\left[\begin{array}{ccc|c} 1 & 0 & 1 & 3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

rref

$x = 3 - t$
$y = 0$
$z = t$

$$\boxed{\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \quad t \in \mathbb{R}}$$

4) Use the adjoint formula for the inverse matrix or Cramer's rule, to solve the following small but messy system of equations:

$$5x + 11y = 2$$
$$7x + 16y = 3$$

(10 points)

Cramer: $x = \dfrac{\begin{vmatrix} 2 & 11 \\ 3 & 16 \end{vmatrix}}{\begin{vmatrix} 5 & 11 \\ 7 & 16 \end{vmatrix}} = \dfrac{32-33}{80-77} = -\dfrac{1}{3}$

check: $-\frac{1}{3}\begin{bmatrix} 5 \\ 7 \end{bmatrix} + \frac{1}{3}\begin{bmatrix} 11 \\ 16 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 6 \\ 9 \end{bmatrix}$

$= \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ ✓

$y = \dfrac{\begin{vmatrix} 5 & 2 \\ 7 & 3 \end{vmatrix}}{3} = \dfrac{15-14}{3} = \dfrac{1}{3}$

or $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 5 & 11 \\ 7 & 16 \end{bmatrix}$

adjoint $A^{-1} = \dfrac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

$= \dfrac{1}{3}\begin{bmatrix} 16 & -11 \\ -7 & 5 \end{bmatrix}$

$\begin{bmatrix} x \\ y \end{bmatrix} = A^{-1}\vec{b} = \frac{1}{3}\begin{bmatrix} 16 & -11 \\ -7 & 5 \end{bmatrix}\begin{bmatrix} 2 \\ 3 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 32-33 \\ -14+15 \end{bmatrix} = \begin{bmatrix} -1/3 \\ 1/3 \end{bmatrix}$ ✓

5) Consider the differential equation

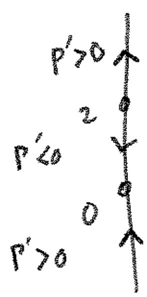
$$\frac{dP}{dt} = P^2 - 2\,P$$

which could be a model for a certain population problem.

5a) Find the equilibrium solutions. (5 points)

$P^2 - 2P = P(P-2)$ $\qquad \boxed{P = 0, 2}$

5b) Sketch the phase portrait for this autonomous DE and decide whether each equilibrium solution is stable or unstable. (5 points)



$P = 0$ stable

$P = 2$ unstable

5c) Which of population models we studied would lead to differential equation of this type? Be as precise as you can, so that you account for the signs of both terms on the right of the differential equation. (5 points)

doomsday-extinction

$P' = aP^2 - bP \qquad a, b > 0$

$= kP(P-M)$

$P > M \rightarrow$ finite time doomsday

$0 < P < M \Rightarrow P \rightarrow 0$ as $t \rightarrow \infty$

extinction

5d) Solve the initial value problem

$$\frac{dP}{dt} = P^2 - 2P = P(P-2)$$
$$P(0) = 3$$

Explain what happens to your solution as time increases, and whether this behavior is consistent with the phase portrait you drew in (5b).

(10 points)

$$\frac{dP}{P(P-2)} = dt$$

$$\frac{1}{2}\left(\frac{1}{P-2} - \frac{1}{P}\right) dP = dt$$

$$\left(\frac{1}{P-2} - \frac{1}{P}\right) dP = 2\,dt$$

int: $\ln\left|\frac{P-2}{P}\right| = 2t + C_1$

exp: $\left|\frac{P-2}{P}\right| = e^{C_1} e^{2t}$

$$\frac{P-2}{P} = Ce^{2t}$$

@ $t=0$, $P_0 = 3$ $\quad \frac{1}{3} = C$

$$\frac{P-2}{P} = \frac{1}{3}e^{2t}$$

$$P-2 = P\left(\frac{1}{3}e^{2t}\right)$$

$$P\left(1 - \frac{1}{3}e^{2t}\right) = 2$$

$$P = \frac{2}{1-\frac{1}{3}e^{2t}}$$

check: $P(0) = \frac{2}{1-\frac{1}{3}} = \frac{2}{2/3} = 3$ ✓.

phase portrait says
P is inc. if $P_0 > 2$,
so this is consistent

denom $= 2/3$ at $t=0$

denom ↘ to zero as

$1 - \frac{1}{3}e^{2t} = 0$

$1 = \frac{1}{3}e^{2t}$

$3 = e^{2t}$

$\ln 3 = 2t$

$t = \frac{\ln 3}{2}$

so,
as $t \to \frac{\ln 3}{2}$
from the right

$P(t) \to \infty$
doomsday!