Math 2250-3
Wed. 8 Sept.

HW due Wed 9/15 (← Note Maple Ia due 9/13, Ib due 9/20 !)

2.1 1, 3, 6, 8, 9, 11, 12, 22, 33
2.2 5, 7, 9, 12, 14 (can also do by hand if you prefer)
15 "
23
2.3 2, 3, 9, 10, 12, 17, 18

- Finish (start!) Torricelli experiment

§2.1 Population growth revisited.

Population $P(t)$ at time t
assume no migration

$\beta(t)$:= normalized birth rate (fertility rate)
= # births per unit time, per unit population
e.g. .1 child/year, per person

so actual birthrate
$$B(t) = \beta(t) \cdot P(t)$$
(child/year)/person · person

$\delta(t)$:= normalized death rate (morbidity)
= # deaths per unit time, per unit population
e.g. .02 deaths/year, per person.

so actual deathrate
$$D(t) = \delta(t) P(t)$$

so $\frac{dP}{dt} = B(t) - D(t)$
$$\boxed{\frac{dP}{dt} = (\beta - \delta) P(t)}$$

1st year Calc model: $\beta = \beta_0$ const, $\delta = \delta_0$ const, so $\frac{dP}{dt} = kP$ with $k = \beta_0 - \delta_0$
exponential growth/decay

logistic model
$$\left.\begin{aligned} \beta &= \beta_0 - \beta_1 P \\ \delta &= \delta_0 + \delta_1 P \end{aligned}\right\}$$ simple corrections to reflect effects of finite resources & competition

yields $$\frac{dP}{dt} = (\beta_0 - \beta_1 P - \delta_0 - \delta_1 P)$$

i.e. $$\frac{dP}{dt} = aP - bP^2 \qquad \begin{pmatrix} a = \beta_0 - \delta_0 \\ b = \beta_1 + \delta_1 \end{pmatrix}$$

or $$\boxed{\frac{dP}{dt} = kP(M-P)} \qquad \begin{pmatrix} k = b \\ M = a/b \end{pmatrix}$$

this is called the logistic model

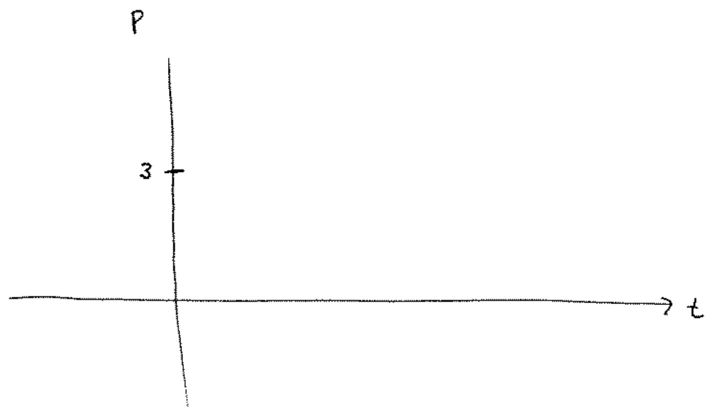
Make slope field for

$$\frac{dP}{dt} = kP(M-P)$$

e.g.

$$\frac{dP}{dt} = 2P(3-P)$$

isoclines = ?



What do you predict for long-time behavior of sol'ns with initial positive populations?

analytic (formulas) solutions : separate variables!

$$\begin{cases} \frac{dP}{dt} = kP(M-P) \\ P(0) = P_0 \end{cases}$$

$$\int \frac{dP}{P(M-P)} = \int k\,dt$$

$$\frac{1}{M}\int \frac{1}{P} + \frac{1}{M-P}\,dP = kt + C \qquad \left(\begin{matrix}\text{Partial} \\ \text{fractions!}\end{matrix}\right)$$

$$\frac{1}{M}\Big[\ln|P| - \ln|M-P|\Big] = kt + C$$

$$\frac{1}{M}\ln\left|\frac{P}{M-P}\right| = kt + C$$

$$\ln\left|\frac{P}{M-P}\right| = Mkt + \tilde{C}$$

$$\left|\frac{P}{M-P}\right| = e^{\tilde{c}}\,e^{Mkt}$$

$$\frac{P}{M-P} = Ce^{Mkt}$$

$P(0) = P_0$, so

$$\frac{P}{M-P} = \frac{P_0}{M-P_0}\,e^{Mkt}$$

$$P = \left(\frac{P_0}{M-P_0}\right)e^{Mkt}(M-P)$$

$$P\left[1 + \frac{P_0}{M-P_0}e^{Mkt}\right] = \frac{MP_0}{M-P_0}\,e^{Mkt}$$

$$P = \frac{\left(\frac{MP_0}{M-P_0}\right)e^{Mkt}}{1 + \left(\frac{P_0}{M-P_0}\right)e^{Mkt}}$$

so,!,

$$\boxed{P(t) = \frac{MP_0}{(M-P_0)e^{-Mkt} + P_0}}$$

long time behavior?

M is called the carrying capacity in logistic models. Why?

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Examples 2,3 from %2.1 of the text, pages 81-83.

The Belgian demographer P.F. Verhulst introduced the logistic model around 1840, as a tool for studying human population growth. Our text demonstrates its superiority to the simple exponential growth model, and also illustrates why mathematical modelers must always exercise care, by comparing the two models to actual U.S. population data. here are actual U.S. populations from 1800-1990, see e.g. the table on page 82:

```
> restart:  #clear Maple memory
> pops:= [[1800,5.3],[1810,7.2],[1820,9.6],[1830,12.9],
          [1840,17.1],[1850,23.2],[1860,31.4],[1870,38.6],
          [1880,50.2],[1890,63.0],[1900,76.2],[1910,92.2],
          [1920,106.0],[1930,123.2],[1940,132.2],[1950,151.3],
          [1960,179.3],[1970,203.3],[1980,225.6],[1990,248.7]]:
```

Unlike Verhulst, the book uses data from 1800, 1850 and 1900 to get constants in our two models. We let t=0 correspond to 1800.

Exponential Model: For the exponential growth model $P(t)=P_0\,e^{(r\,t)}$ we use the 1800 and 1900 data to get values for P_0 and r:

```
> P0:=5.308;
  solve(P0*exp(r*100)=76.212,r);
```

$$P0 := 5.308$$

$$0.02664303814$$

```
> P1:=t->5.308*exp(.02664*t);  #exponential model
```

$$P1 := t \to 5.308\,e^{(0.02664\,t)}$$

Logistic Model: We get P_0 from 1800, and use the 1850 and 1900 data to find k and M:

```
> P2:=t->M*P0/(P0+(M-P0)*exp(-M*k*t));
     #logisitic function, with our P0
```

$$P2 := t \to \frac{M\,P0}{P0+(M-P0)\,e^{(-M\,k\,t)}}$$

```
> solve({P2(50)=23.192, P2(100)=76.212},{M,k});
```

$$\{k=0.0001677157274,\ M=188.1208275\}$$

```
> M:=188.1208275;
  k:=.1677157274e-3;
  P2(t);  #should be our logistic model function,
     #equation (11) page 82.
```

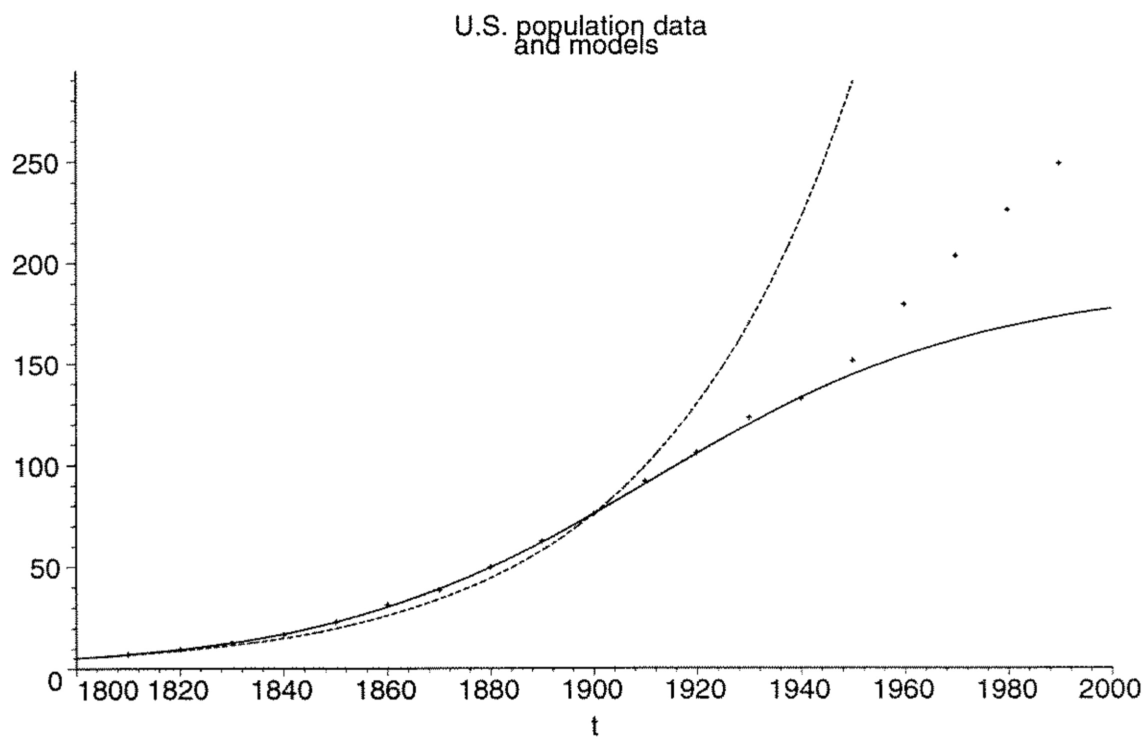
$$M := 188.1208275$$

$$k := 0.0001677157274$$

$$\frac{998.5453524}{5.308+182.8128275\,e^{(-0.03155082142\,t)}}$$

Now compare the two models with the real data, and discuss:

```
> with(plots):
  plot1:=plot(P1(t-1800),t=1800..1950,color=black, linestyle=3):
    #this linestyle gives dashes for the exponential curve
  plot2:=plot(P2(t-1800),t=1800..2000,color=black):
  plot3:=pointplot(pops,symbol=cross):
  display({plot1,plot2,plot3},title=`U.S. population data
  and models`);
Warning, the name changecoords has been redefined
```




```
>
```

The exponential model takes no account of the fact that the U.S. has only finite resources. Any ideas on why the logistic model begins to fail (with our parameters) around 1950?