Math 2250-3
Monday Sept 27

§3.5 Inverse matrices

identity matrix $I = \begin{bmatrix} 1 & 0 & & 0 \\ 0 & 1 & & \\ & 0 & \ddots & \\ 0 & 0 & & 1 \end{bmatrix}$

notice $col_j(I) = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \leftarrow$ entry j $:= \vec{e}_j$

$row_i(I) = [0, 0, \ldots 0, 1, 0 \ldots 0] = \vec{e}_i$
↑
entry i

$A_{m\times n} I_{n\times n} = A$

check: $entry_{ij}(AI) = row_i(A)\cdot col_j(I) = row_i(A)\cdot \vec{e}_j = a_{ij}$ ✓

$I_{m\times m} A_{m\times n} = A$

check: $entry_{ij}\, IA = row_i(I)\cdot col_j(A) = \vec{e}_i \cdot col_j(A) = a_{ij}$ ✓

Def $A_{n\times n}$ is invertible $:=$ there is a matrix B s.t. $AB = BA = I$
(and we write A^{-1} for B)

[A can have at most 1 inverse since if $AB = BA = I$
& $AC = CA = I$
then $CAB = C(AB) = CI = C$
$\quad\; \| \; (CA)B = IB = B$ so $B = C$.]

Example $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ $\quad A^{-1} = \begin{bmatrix} -2 & 1 \\ +3/2 & -1/2 \end{bmatrix}$

check:

What good is A^{-1}?

<u>Theorem</u> If A^{-1} exists then the only solution to $A\vec{x}=\vec{b}$ is
$\vec{x}=A^{-1}\vec{b}$

<u>example cont'd</u>

solve
$$x+2y=5$$
$$3x+4y=6$$

$$\begin{bmatrix}1&2\\3&4\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}5\\6\end{bmatrix}\quad;\quad\begin{bmatrix}x\\y\end{bmatrix}=\underbrace{\begin{bmatrix}-2&1\\3/2&-1/2\end{bmatrix}}_{A^{-1}}\begin{bmatrix}5\\6\end{bmatrix}=\begin{bmatrix}-4\\9/2\end{bmatrix}$$

check:

check:

But how did I know A^{-1} in that example?
<u>Ans</u>: I solved the matrix eqtn

$$AX = \text{Id.}$$

for $\text{col}_1(X)$: $\left[\begin{array}{cc|c}1&2&1\\3&4&0\end{array}\right]$

for $\text{col}_2(X)$: $\left[\begin{array}{cc|c}1&2&0\\3&4&1\end{array}\right]$

for both columns at once:

$$\left[\begin{array}{cc|c|c}1&2&1&0\\3&4&0&1\end{array}\right]$$

$$\begin{array}{l}\\-3R_1+R_2\end{array}\left[\begin{array}{cc|c|c}1&2&1&0\\0&-2&-3&1\end{array}\right]$$

$$\begin{array}{l}R_1+R_2\\\\\end{array}\left[\begin{array}{cc|c|c}1&0&-2&1\\0&-2&-3&1\end{array}\right]$$

$$\begin{array}{l}\\-\frac{R_2}{2}\end{array}\left[\begin{array}{cc|c|c}1&0&-2&1\\0&1&3/2&-1/2\end{array}\right]$$

$\uparrow$ col_1 $\uparrow$ col_2

so $\begin{bmatrix}1&2\\3&4\end{bmatrix}\begin{bmatrix}-2&1\\3/2&-1/2\end{bmatrix}=\begin{bmatrix}1&0\\0&1\end{bmatrix}$.

This is a general algorithm!

The matrix equation for $X = A^{-1}$ is

$$AX = I \qquad A_{n\times n},\ X_{n\times n},\ I_{n\times n}$$

Synthetically this is the n by $2n$ augmented matrix

$$A \,\vdots\, I$$

<u>If</u> this reduces to rref equal to

$$I \,\vdots\, B$$

Then $X = B$, so $AB = I$

[notice if you write $B \,\vdots\, I$ and do the row ops backwards you get $I \,\vdots\, A$, so also $BA = I$ is automatic].

so

<u>Thm</u> if $\mathrm{rref}(A) = I$ then A^{-1} exists and can be found by algorithm above.

<u>Thm</u> if A^{-1} exists then unique sol'n to $A\vec{x} = \vec{b}$ is $\vec{x} = A^{-1}\vec{b}$

so, if A^{-1} exists we must also have $\mathrm{rref}(A) = I$! $\quad$ ($\mathrm{rref}(A) \neq I \Rightarrow$ never get unique sol'tns).

$$\boxed{A^{-1} \text{ exists iff } \mathrm{rref}(A) = \text{id. iff } A\vec{x} = \vec{b} \text{ always has unique sol'tn}}$$

<u>example</u> $\quad A = \begin{bmatrix} 1 & 5 & 1 \\ 2 & 5 & 0 \\ 2 & 7 & 1 \end{bmatrix}$

$$\left[\begin{array}{ccc|ccc} 1 & 5 & 1 & 1 & 0 & 0 \\ 2 & 5 & 0 & 0 & 1 & 0 \\ 2 & 7 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\begin{array}{r} \\ -2R_1+R_2 \\ -2R_1+R_3 \end{array}\left[\begin{array}{ccc|ccc} 1 & 5 & 1 & 1 & 0 & 0 \\ 0 & -5 & -2 & -2 & 1 & 0 \\ 0 & -3 & -1 & -2 & 0 & 1 \end{array}\right]$$

$$\begin{array}{r} R_2+R_1 \\ \\ -R_3 \end{array}\left[\begin{array}{ccc|ccc} 1 & 0 & -1 & -1 & 1 & 0 \\ 0 & -5 & -2 & -2 & 1 & 0 \\ 0 & 3 & 1 & 2 & 0 & -1 \end{array}\right]$$

$$\begin{array}{r} \\ 2R_3+R_2 \\ \\ \end{array}\left[\begin{array}{ccc|ccc} 1 & 0 & -1 & -1 & 1 & 0 \\ 0 & 1 & 0 & 2 & 1 & -2 \\ 0 & 3 & 1 & 2 & 0 & -1 \end{array}\right]$$

$$\begin{array}{r} \\ \\ -3R_2+R_3 \end{array}\left[\begin{array}{ccc|ccc} 1 & 0 & -1 & -1 & 1 & 0 \\ 0 & 1 & 0 & 2 & 1 & -2 \\ 0 & 0 & 1 & -4 & -3 & 5 \end{array}\right]$$

$$\begin{array}{r} R_3+R_1 \\ \\ \\ \end{array}\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -5 & -2 & 5 \\ 0 & 1 & 0 & 2 & 1 & -2 \\ 0 & 0 & 1 & -4 & -3 & 5 \end{array}\right]$$

$$A^{-1} = \begin{bmatrix} -5 & -2 & 5 \\ 2 & 1 & -2 \\ -4 & -3 & 5 \end{bmatrix}$$

Nice formula for inverse of 2×2 matrix:

$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1}$ exists iff $D = ad-bc \neq 0$,

and in this case, it equals

$$\frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$ (See example 1)

how did we find this?

$ad-bc\neq 0$ is exactly the condition that row_1 and row_2 are not multiples, i.e. $\text{rref}(A) = I$.

do row ops with letters, or check the formula by computing AA^{-1}.

$$\left[\begin{array}{cc|cc} a & b & 1 & 0 \\ c & d & 0 & 1 \end{array}\right]$$

$\frac{1}{a}R_1$ $$\left[\begin{array}{cc|cc} 1 & b/a & 1/a & 0 \\ c & d & 0 & 1 \end{array}\right]$$ [if $a\neq 0$]

$-cR_1+R_2$ $$\left[\begin{array}{cc|cc} 1 & b/a & 1/a & 0 \\ 0 & d-\frac{cb}{a} & -\frac{c}{a} & 1 \end{array}\right]$$

common denom $$\left[\begin{array}{cc|cc} 1 & b/a & 1/a & 0 \\ 0 & \frac{D}{a} & -\frac{c}{a} & 1 \end{array}\right]$$

$\frac{a}{D}R_2$ $$\left[\begin{array}{cc|cc} 1 & b/a & 1/a & 0 \\ 0 & 1 & -\frac{c}{D} & \frac{a}{D} \end{array}\right]$$ [if $D\neq 0$]

$-\frac{b}{a}R_2+R_1$ $$\left[\begin{array}{cc|cc} 1 & 0 & \frac{1}{a}+\frac{bc}{aD} & -b/D \\ 0 & 1 & -c/D & a/D \end{array}\right]$$ $\frac{D+bc}{aD} = \frac{d}{D}$

$$\left[\begin{array}{cc|cc} 1 & 0 & \frac{d}{D} & -\frac{b}{D} \\ 0 & 1 & -\frac{c}{D} & \frac{a}{D} \end{array}\right]$$

$$A^{-1} = \frac{1}{D}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \quad !$$