Math 2250-3
September 22, 2004
Maple linear algebra

```
> with(linalg): #load linear algebra package
```

Coefficient matrix taken from problem #19, section 3.3, page 170. (You are assigned #20).

```
> A:= matrix(3,5,[2,7,-10,-19,13,1,3,-4,-8,6,1,0,2,1,3]);
      #the coefficient matrix
```

$$A := \begin{bmatrix} 2 & 7 & -10 & -19 & 13 \\ 1 & 3 & -4 & -8 & 6 \\ 1 & 0 & 2 & 1 & 3 \end{bmatrix}$$

```
> rref(A);  #compute the reduced row echelon form
```

$$\begin{bmatrix} 1 & 0 & 2 & 1 & 3 \\ 0 & 1 & -2 & -3 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

I want to consider three different linear systems for which A is the coefficient matrix. In the first one, the right hand sides are all zero, and I have carefully picked the other two right hand sides (by working backwards from rref(A), actually). The three right hand sides are the columns of the matrix B:

```
> B:=matrix(3,3,[0,7,7,0,0,3,0,0,0]);
```

$$B := \begin{bmatrix} 0 & 7 & 7 \\ 0 & 0 & 3 \\ 0 & 0 & 0 \end{bmatrix}$$

We can consider all three linear systems at once by augmenting B to A, and then rref-ing: (you'll have to put in the vertical line separating the coefficient matrix from the right hand sides by yourself.)

```
> C:=augment(A,B);
  rrefC:=rref(C);
```

$$C := \begin{bmatrix} 2 & 7 & -10 & -19 & 13 & 0 & 7 & 7 \\ 1 & 3 & -4 & -8 & 6 & 0 & 0 & 3 \\ 1 & 0 & 2 & 1 & 3 & 0 & 0 & 0 \end{bmatrix}$$

$$rrefC := \begin{bmatrix} 1 & 0 & 2 & 1 & 3 & 0 & 0 & 0 \\ 0 & 1 & -2 & -3 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

From rref(C) you can read off the solutions to each of the three problems. Note that each column (from rref(A)) without a leading 1 in it gives you a free parameter in the solution. Write down the solutions to our three problems on the next page:

```
> rref(C);
```

$$\begin{bmatrix} 1 & 0 & 2 & 1 & 3 & 0 & 0 & 0 \\ 0 & 1 & -2 & -3 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

Important conceptual questions:

(1) Which of these three solutions could you have written down just from rrefA), rather than from the rref of the augmented matrix? Why?

(2) Linear systems in which right hand side vector equals zero are called homogeneous linear systems. Otherwise they are called inhomogeneous. Notice that the general solution to the consistent inhomogeneous system is the sum of a particular solution to it, together with the general solution to the homogeneous system!!! Was this an accident? We'll come back to this.

By the way, Maple will go ahead and solve a linear system directly if you ask it to. Here's the consistent inhomogeneous problem:

```
> b:=vector([7,3,0]);
```

$$b := [7, 3, 0]$$

```
> linsolve(A,b);
```

$$[-2\,\_t_1 - \_t_2 - 3\,\_t_3,\ 2\,\_t_1 + 3\,\_t_2 - \_t_3 + 1,\ \_t_1,\ \_t_2,\ \_t_3]$$

Is this the answer we got?

Using rrref(A) to discern general facts about the solutions to Ax=b:

The reduced row echelon form of a matrix tells you a lot about possible solutions to the matrix equation Ax=b. What can you say in the following situations?

```
> AA:=matrix(2,5,[2, 7, -10, -19, 13,1, 3, -4, -8, 6]);
  rref(AA);
```

$$AA := \begin{bmatrix} 2 & 7 & -10 & -19 & 13 \\ 1 & 3 & -4 & -8 & 6 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 2 & 1 & 3 \\ 0 & 1 & -2 & -3 & 1 \end{bmatrix}$$

Is the homogeneous problem Ax=0 always solvable?

Is the inhomogeneous problem Ax=b always solvable?

When it is solvable, how many solutions are there?

```
> B:=matrix(3,2,[1,2,-1,3,4,2]);
```

$$B := \begin{bmatrix} 1 & 2 \\ -1 & 3 \\ 4 & 2 \end{bmatrix}$$

```
> RREFB:=rref(B);
```

$$RREFB := \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

How many solutions to the homogeneous problem Bx=0?

Is the inhomogeneous problem Bx=b always solvable?

When it is solvable, how many solutions does it have?

```
> C:=matrix(4,4,[1,0,-1,1,22,-1,3,5,7,4,6,2,3,5,7,13]);
```

$$C := \begin{bmatrix} 1 & 0 & -1 & 1 \\ 22 & -1 & 3 & 5 \\ 7 & 4 & 6 & 2 \\ 3 & 5 & 7 & 13 \end{bmatrix}$$

```
> RREFC:=rref(C);
```

$$RREFC := \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Square matrices with 1's down the diagonal (which runs from the upper left to lower right corner) are special. They are called identity matrices.

How many solutions to the homogeneous problem Cx=0?

Is the inhomogeneous problem Cx=b always solvable?

How many solutions?

What are your general conclusions?

(1) What conditions on rref(A) guarantee that the homogeneous equation Ax=0 has infinitely many solutions?

(2) What conditions on the dimensions of A always force infinitely many solutions to the homogeneous problem regardless of the individual entries of A?

(3) What conditions on rref(A) guarantee that solutions x to Ax=b are always unique (if they exist)?

(4) If A is a square matrix, what can you say about solutions to Ax=b when
(4a) rref(A) is the identity matrix

(4b) rref(A) is not the identity matrix?

Math 2250-3
Wed 9/22

HW for 9/29

3.3 13, 20, 33, 34
3.4 3, 5, 7, 10, 13, 19, 31 32, 34, 39, 40, 44
3.5 5, 7 8, 13, 22 23 30 31 32 33

Maple (or Matlab) checks : (to be handed in)

3.3 20 → reduced row echelon form
3.4 13
3.5 22 → find A^{-1} & verify $AA^{-1} = A^{-1}A = I$
23 → there is a command which does this, or you can use rref.

You create the document and use "help" button to find commands.
(hint: a lot of them also appear in class handouts)

rewrite

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m \end{aligned}$$

as
$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \cdot & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

i.e. $$A\vec{x} = \vec{b}$$

A = coeff matrix
$\vec{b}$ = RHS vector

properties of matrix times vector:

$$\text{entry}_i(A\vec{x}) = \text{row}_i(A) \cdot \vec{x} \qquad \text{(dot product)}$$

recall $\vec{u} = \begin{bmatrix} u_1 \\ u_2 \\ \\ u_n \end{bmatrix}, \vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ \cdot \\ v_n \end{bmatrix}$ $\vec{u} \cdot \vec{v} = u_1v_1 + u_2v_2 + \cdots + u_nv_n = \sum u_iv_i$

$$\vec{u} \cdot [\vec{v} + \vec{w}] = \vec{u} \cdot \vec{v} + \vec{u} \cdot \vec{w}$$
$$\vec{u} \cdot [k\vec{v}] = (k\vec{u} \cdot \vec{v}) = k\vec{u} \cdot \vec{v}$$

so
$$A[\vec{x} + \vec{y}] = A\vec{x} + A\vec{y}$$
$$A[k\vec{x}] = kA\vec{x}$$

Return to question from page 2:

general soltn to

$$A\vec{x} = \vec{b}$$

is of form $\vec{x} = \underset{\substack{\uparrow \\ \text{particular} \\ \text{soltn}}}{\vec{x}_p} + \underset{\substack{\nwarrow \\ \text{general soltn} \\ \text{to homog} \\ \text{problem.}}}{\vec{x}_H}$

proof If $A\vec{x}_p = \vec{b}$ and $A\vec{x}_H = \vec{0}$

then
$$\begin{aligned} A(\vec{x}_p + \vec{x}_H) &= A\vec{x}_p + A\vec{x}_H \\ &= \vec{b} + \vec{0} \\ &= \vec{b} \end{aligned}$$

so $\vec{x} = \vec{x}_p + \vec{x}_H$ is always a soltn.

If $\vec{x}$ is any soltn to $A\vec{x} = \vec{b}$

then $\vec{x} = \vec{x}_p + (\vec{x} - \vec{x}_p)$

so $A\vec{x} = A[\vec{x}_p + (\vec{x} - \vec{x}_p)]$

$= A\vec{x}_p + A(\vec{x} - \vec{x}_p)$

so $\vec{b} = \vec{b} + A(\vec{x} - \vec{x}_p)$

so $0 = A(\vec{x} - \vec{x}_p)$

so $\vec{x} - \vec{x}_p = \vec{x}_H$

$\vec{x} = \vec{x}_p + \vec{x}_H$.

"same" as

$L(y) := y' + p(x)y$

$L(y_1 + y_2) = L(y_1) + L(y_2)$

$L(cy) = cL(y)$

Check!

So same conclusion holds,

$y = y_p + y_H$.

example:

$y' + 3y = 6$

$e^{3x}[\underset{\substack{\shortparallel \\ (e^{3x}y)'}}{y' + 3y}] = 6e^{3x}$

$e^{3x}y = 2e^{3x} + C$

$y = \underset{\substack{\uparrow \\ y_p}}{2} + \underset{\substack{\uparrow \\ y_H}}{Ce^{-3x}}$!!