Math 2250-3
Monday 9/13
§ 2.2 & 2.3

example 4 from 2.2, page 94, is interesting

<u>logistic equation w/ harvesting</u> [think fish, deer etc.]

$$\frac{dx}{dt} = \underbrace{kx(M-x)}_{\text{logistic}} \underbrace{-h}_{\text{constant-rate harvesting}}$$

~ not the same as in § 2.2 #23 HW.

$$\frac{dx}{dt} = -kx^2 + kMx - h$$

$$= -k\underbrace{\left(x^2 - Mx + \frac{h}{k}\right)}_{(x-N)(x-H)}$$

$$\frac{dx}{dt} = -k(x-N)(x-H)$$

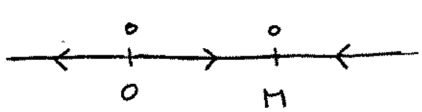
$$H, N = \frac{M \pm \sqrt{M^2 - 4h/k}}{2}$$

take $N = \dfrac{M + \sqrt{M^2 - 4h/k}}{2}$

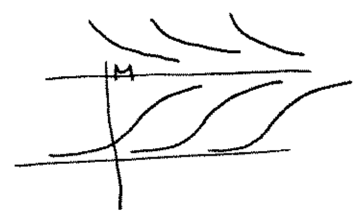
$H = \dfrac{M - \sqrt{M^2 - 4h/k}}{2}$

<u>phase portraits</u>

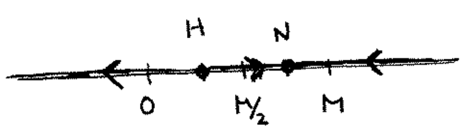
$h = 0$ (logistic)

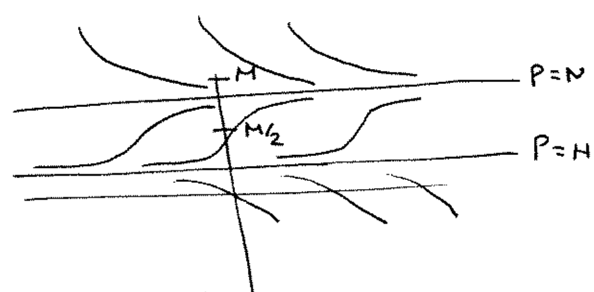


slope field

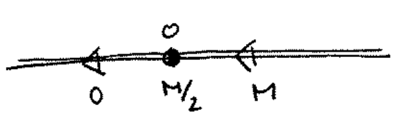


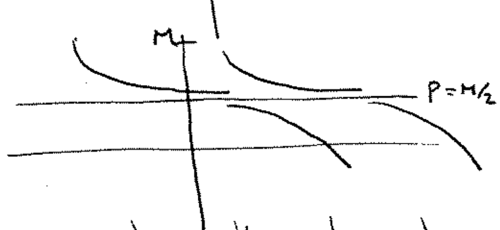
$0 < h < \frac{kM^2}{4}$
$(M^2 - \frac{4h}{k} > 0)$
$M > N > M/2$
$0 < H < M/2$



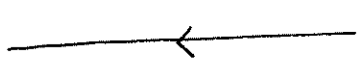


critical harvesting
$h = \frac{kM^2}{4}$

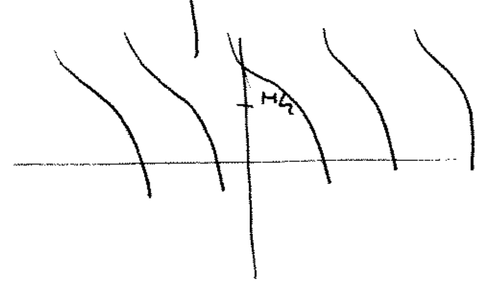




post-critical:
$h > \frac{kM^2}{4}$



interpret!



§2.3 velocity / acceleration models

In calc:

$\uparrow y$ $\qquad m\frac{dv}{dt} = F_G = -g$

$$\frac{dv}{dt} = -g$$

$$v = -gt + v_0$$

$$y = -\tfrac{1}{2}gt^2 + v_0 t + y_0$$

add resistance?

$$m\frac{dv}{dt} = F_G + F_f$$

$$|F_f| \cong k|v|^p \qquad 1 \le p \le 2 \quad \text{empiricle}$$

$\underline{p=1}$ ("linear" model)

$$m\frac{dv}{dt} = -mg - kv$$

force will cause an acceleration opposite to direction of motion

... this is low speed "linearization"

if $F_f = F(v) = F(0) + F'(0)v + \frac{1}{2!}F''(0)v^2 + \dots$, where $F(0) = 0$

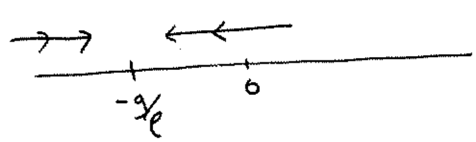
for $|v|$ small, "negligable"

$$\frac{dv}{dt} = -g - \rho v \qquad \rho := \frac{k}{m}$$

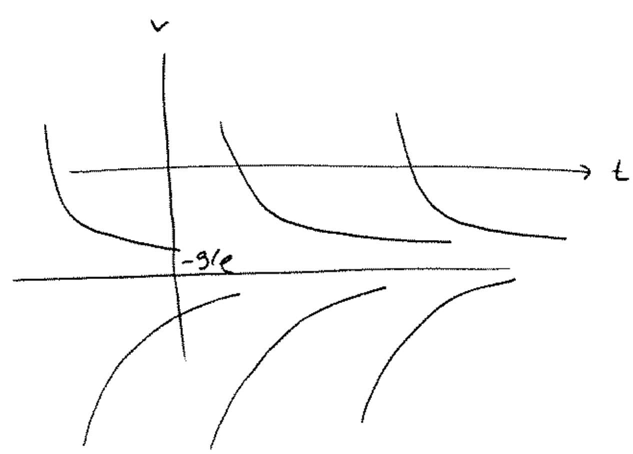
$$= -\rho\left(v + \frac{g}{\rho}\right)$$

equil soltn: $v = -g/\rho$

phase portrait:



slope field:



analytic sol'n

$$\frac{dv}{dt} = -\rho\left(v + \frac{g}{\rho}\right)$$

$$\frac{dv}{v + g/\rho} = -\rho\, dt$$

$$\ln|v + g/\rho| = -\rho t + C \qquad t = 0 \Rightarrow C = \ln|v_0 + g/\rho|$$

$$= -\rho t + \ln|v_0 + g/\rho|$$

$$|v + g/\rho| = |v_0 + g/\rho|\, e^{-\rho t}$$

$$v + g/\rho = (v_0 + g/\rho)\, e^{-\rho t} \qquad \text{(why?)}$$

$$\boxed{v = \underbrace{-g/\rho}_{v_T \text{ "terminal velocity"}} + \underline{(v_0 + g/\rho)\, e^{-\rho t}}}$$

$$y = \int v(t)\,dt = t v_T + \left(\frac{v_0 - v_T}{-\rho}\right) e^{-\rho t} + C \qquad y_0 = \frac{v_T - v_0}{\rho} + C$$

$$\text{so } C = y_0 + \frac{v_0 - v_T}{\rho}$$

$$\boxed{y = t v_T + \left(\frac{v_0 - v_T}{\rho}\right)\left(1 - e^{-\rho t}\right) + y_0}$$

Examples 1 & 2 p. 98-100

crossbow bolt

$v_0 = 49$ m/s

$g = 9.8$ m/s²

$y_0 = 0$

no friction

$y = -4.9t^2 + 49t$

$v = -9.8t + 49$

max ht at $t = 5$ sec

$y_{max} = y(5) = 49(2.5)$

$= 122.5$ m

time aloft = 10 sec.

linear drag

$\rho = .04$ (drag coeff; empirical)

corresponds to

$|v_T| = \frac{g}{\rho} = \frac{9.8}{.04} = 245$ m/sec

(you could measure this to deduce ρ)

$v_0 - v_T = 49 + 245 = 294$

so

$v = -245 + 294\, e^{-.04t}$

$v = 0$ at $\frac{245}{294} = e^{-.04t}$

$t_{max} \approx 4.56$ sec

$y = -245t + (294)(25)(1 - e^{-.04t})$

$y(t_{max}) = ?$

When does bolt hit ground?

2250 -3, Monday September 8
Computation sheet for Example 2, section 2.3

Time of maximum height:

```
> 25*ln(294.0/245);
  solve(-245+294*exp(-.04*t)=0,t); #should be same
```

4.558038920

4.558038920

Formulas for height and velocity functions

```
> v:= t -> 294*exp(-.04*t) - 245;
```

$$v := t \to 294\, e^{(-0.04\, t)} - 245$$

```
> y:= t-> -245*t + 294*25*(1 - exp(-.04*t));
```

$$y := t \to -245\, t + 7350 - 7350\, e^{(-0.04\, t)}$$

Maximum height, return time tor ground time spent falling, landing speed:

```
> y(4.558038920); #max height
```

108.280465

```
> solve(y(t)=0,t);  #find when returns to ground
```

9.410949931, 0.

```
> 9.410949931 - 4.558038920; #time descending
```

4.852911011

```
> v(9.410949931); #speed when it lands
```

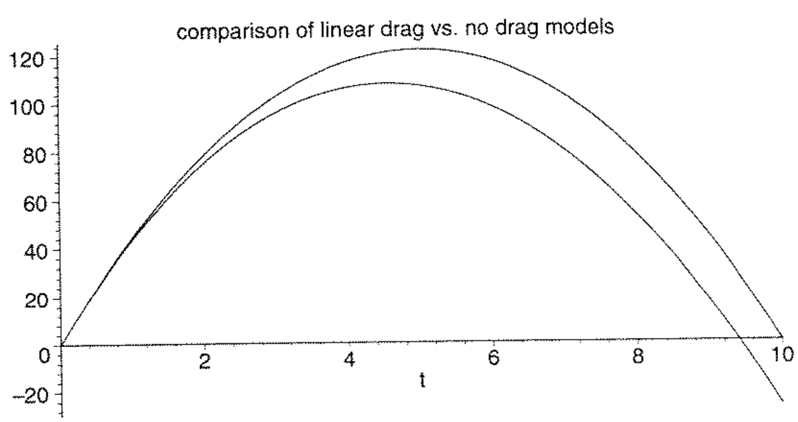
-43.2273093

Conclusions: bolt rises for 4.56 seconds, to a height of 108.3 meters. Then it spends 4.85 seconds descending, landing with a velocity of -43.3 meters per second.

```
> with(plots):
Warning, the name changecoords has been redefined
> z:= t->-4.9*t^2 + 49*t;  #the no resistance model
```

$$z := t \to -4.9\, t^2 + 49\, t$$

```
> plot({z(t),y(t)}, t = 0..10, color=black,
  title='comparison of linear drag vs. no drag models');
```



quadratic drag is also interesting...

going up:

$$m\frac{dv}{dt} = -mg - kv^2$$

$$\frac{dv}{dt} = -g\left(1+\frac{f}{g}v^2\right)$$

$$\frac{dv}{1+\frac{f}{g}v^2} = -g\,dt \;..$$

arctan!

going down:

$$m\frac{dv}{dt} = -mg + kv^2$$

$$\frac{dv}{dt} = -g\left(1-\frac{f}{g}v^2\right)$$

$$\frac{dv}{1-\frac{f}{g}v^2} = -g\,dt$$

parfrac! (or $\tanh^{-1}$)