Math 2250-3
Wed Oct 6
§ 4.1 - 4.3
(to be continued)

HW for Wed 10/13

4.1 1 7 9 10 15 16 19 22 25 26 31 33
4.2 5, 6 9 15 18 24 27 29
4.3 1 3 6 9 10

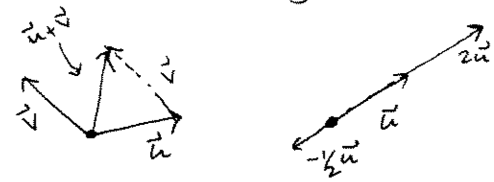
If you need rref for a problem, feel free to compute it with technology

In this chapter we will continuously make use of the "linear combination form" way of writing a matrix times a vector:

chapter 3 chapter 4

$$\begin{bmatrix} a_{11} & a_{12} \cdots & a_{1n} \\ a_{21} & a_{22} \cdots & a_{2n} \\ \vdots & & \\ a_{m1} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n \\ \vdots \\ a_{m1}x_1 + \dots + a_{mn}x_n \end{bmatrix} = x_1 \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix} + x_2 \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix} + \dots + x_n \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix}$$

and you will need to remember the geometric meaning of vector addition and scalar multiplication.



__example__

- How can you get to $\begin{bmatrix} 1 \\ 8 \end{bmatrix}$ in $\mathbb{R}^2$ by only moving in the $\begin{bmatrix} 3 \\ -2 \end{bmatrix}$ and $\begin{bmatrix} 5 \\ 1 \end{bmatrix}$ directions?

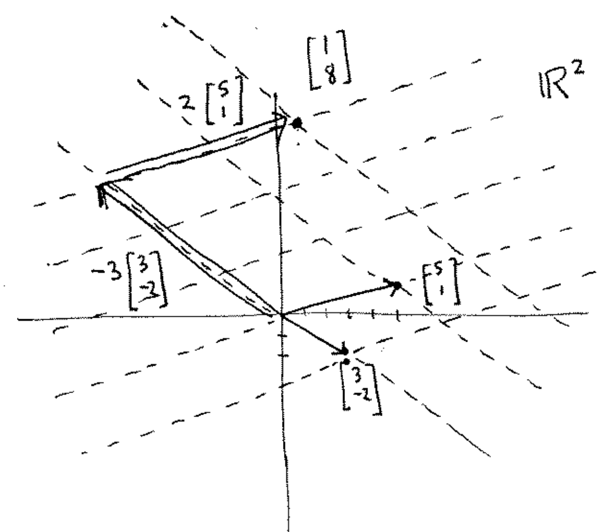
Solve

$$s\begin{bmatrix} 3 \\ -2 \end{bmatrix} + t\begin{bmatrix} 5 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 8 \end{bmatrix}$$

$$\begin{bmatrix} 3 & 5 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} s \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ 8 \end{bmatrix}$$

$$\begin{bmatrix} s \\ t \end{bmatrix} = \frac{1}{13}\begin{bmatrix} 1 & -5 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 8 \end{bmatrix}$$

$$= \frac{1}{13}\begin{bmatrix} -39 \\ 26 \end{bmatrix} = \begin{bmatrix} -3 \\ 2 \end{bmatrix}$$



the expression $s\begin{bmatrix} 3 \\ -2 \end{bmatrix} + t\begin{bmatrix} 5 \\ 1 \end{bmatrix}$ is called a <u>linear combination</u> of $\begin{bmatrix} 3 \\ -2 \end{bmatrix}$ and $\begin{bmatrix} 5 \\ 1 \end{bmatrix}$

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- Can we get anywhere in $\mathbb{R}^2$ via a linear combination of $\begin{bmatrix} 3 \\ -2 \end{bmatrix}$ and $\begin{bmatrix} 5 \\ 1 \end{bmatrix}$?
- Are the linear combination coefficients (s & t) unique, if $s\begin{bmatrix} 3 \\ -2 \end{bmatrix} + t\begin{bmatrix} 5 \\ 1 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$?

example:

$$\vec{i} = \vec{e}_1 = \begin{bmatrix}1\\0\\0\end{bmatrix}, \quad \vec{j} = \vec{e}_2 = \begin{bmatrix}0\\1\\0\end{bmatrix}, \quad \vec{k} = \begin{bmatrix}0\\0\\1\end{bmatrix} = \vec{e}_3$$

these 3 vectors are often called the standard basis vectors for $\mathbb{R}^3$

~~exam~~

$$\begin{bmatrix}7\\-1\\3\end{bmatrix} = 7\vec{e}_1 - \vec{e}_2 + 3\vec{e}_3$$

- Can we express any vector in $\mathbb{R}^3$ as a linear combination $c_1\vec{e}_1 + c_2\vec{e}_2 + c_3\vec{e}_3$ of $\{\vec{e}_1, \vec{e}_2, \vec{e}_3\}$?
- Is this expression unique? (i.e. is there exactly 1 such expression?)

example:

$$\vec{v}_1 = \begin{bmatrix}2\\1\\1\end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix}-1\\1\\-5\end{bmatrix}, \quad \vec{v}_3 = \begin{bmatrix}4\\-1\\11\end{bmatrix}$$

- is $\begin{bmatrix}6\\3\\3\end{bmatrix}$ a linear combination of $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$?
- is this expression (i.e. the linear combo coefficients) unique?

$$c_1\begin{bmatrix}2\\1\\1\end{bmatrix} + c_2\begin{bmatrix}-1\\1\\-5\end{bmatrix} + c_3\begin{bmatrix}4\\-1\\11\end{bmatrix} = \begin{bmatrix}6\\3\\3\end{bmatrix}$$

$$\left[\begin{array}{ccc|c} 2 & -1 & 4 & 6 \\ 1 & 1 & -1 & 3 \\ 1 & -5 & 11 & 3 \end{array}\right] \xrightarrow{\text{rref}} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 3 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$c_3 = t$
$c_2 = 2t$
$c_1 = 3 - t$; $\begin{bmatrix}c_1\\c_2\\c_3\end{bmatrix} = \begin{bmatrix}3\\0\\0\end{bmatrix} + t\begin{bmatrix}-1\\2\\1\end{bmatrix}$

so, e.g.

$$\begin{bmatrix}6\\3\\3\end{bmatrix} = 3\vec{v}_1$$

or $\begin{bmatrix}6\\3\\3\end{bmatrix} = 2\vec{v}_1 + 2\vec{v}_2 + \vec{v}_3 \quad (t=1)$

or $\begin{bmatrix}6\\3\\3\end{bmatrix} = (3-t)\vec{v}_1 + 2t\vec{v}_2 + t\vec{v}_3$ any t

- Can we express any point in $\mathbb{R}^3$ as a linear combo of $\vec{v}_1, \vec{v}_2, \vec{v}_3$?
- If $\vec{b}$ is a linear combo of $\vec{v}_1, \vec{v}_2, \vec{v}_3$, are the linear combo coefficients unique?

Suppose we have three vectors $\vec{v}_1, \vec{v}_2, \vec{v}_3$ in $\mathbb{R}^3$ (not necessarily the ones in the last example)

Think of at least 2 ways to check whether

(i) every vector $\vec{w}$ in $\mathbb{R}^3$ can be expressed as a linear combo of $\vec{v}_1, \vec{v}_2, \vec{v}_3$:

$$\vec{w} = c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3$$

(ii) c_1, c_2, c_3 are unique.

Generalization to $\mathbb{R}^n$?

What if you have $< n$ vectors in $\mathbb{R}^n$

or $> n$ vectors in $\mathbb{R}^n$?

If $\{\vec{v}_1, \ldots \vec{v}_k\}$ is a collection of vectors (say in $\mathbb{R}^n$)
then

$\vec{w}$ is a <u>linear combination</u> of $\{\vec{v}_1, \ldots \vec{v}_k\}$ means

$$\vec{w} = c_1\vec{v}_1 + c_2\vec{v}_2 + \ldots + c_k\vec{v}_k$$ for some choice of <u>linear combination coefficients</u> $c_1, c_2, \ldots c_n$

The <u>span</u> of $\{\vec{v}_1, \ldots \vec{v}_k\}$ is the collection of all $\vec{w}$'s which can be expressed as linear combinations of $\vec{v}_1 - \vec{v}_k$, i.e.

$$\vec{w} = c_1\vec{v}_1 + \ldots + c_n\vec{v}_n$$

<u>example</u>: In the second example of page 2 the span of $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ was not all of $\mathbb{R}^3$. Rather it was a plane thru the origin, namely the span of $\{\vec{v}_1, \vec{v}_2\}$.

(since $\vec{v}_3 = \vec{v}_1 - 2\vec{v}_2$).

If, as above, every $\vec{w}$ in the span of $\{\vec{v}_1, - \vec{v}_k\}$ can be <u>uniquely</u> expressed as a linear combination of $\vec{v}_1 .. \vec{v}_k$ then we call $\{\vec{v}_1, - \vec{v}_k\}$ <u>linearly independent</u> (Otherwise we call them <u>dependent</u>)

Here's why:

If some $\vec{w} = c_1\vec{v}_1 + \ldots + c_k\vec{v}_k$

$\vec{w} = d_1\vec{v} + \ldots + d_k\vec{v}_k$ not all d_i's $= c_i$'s.

this is true if and only if

$$\vec{w} - \vec{w} = \vec{O} = (c_1 - d_1)\vec{v}_1 + \cdots + (c_k - d_k)\vec{v}_k \qquad \text{not all } c_i - d_i = 0$$

i.e.

how we check for linear independence

$$\vec{O} = \tilde{c}_1\vec{V}_1 + \tilde{c}_2\vec{V}_2 + \ldots + \tilde{c}_k\vec{v}_k \qquad \text{not all } \tilde{c}_i = 0$$

and this is the same as (picking some $\tilde{c}_j \neq 0$)

$$-\tilde{c}_j\vec{V}_j = \tilde{c}_1\vec{V}_1 + \ldots + \tilde{c}_{j-1}\vec{V}_{j-1} + \tilde{c}_{j+1}\vec{V}_{j+1} + \ldots + c_k\vec{v}_k$$

$$-\tfrac{1}{\tilde{c}_j}(\qquad \vec{V}_j = k_1\vec{V}_1 + \ldots + k_{j-1}\vec{V}_{j-1} + k_{j+1}\vec{V}_{j+1} + \ldots + c_k\vec{v}_k$$

saying that at least one of the $\vec{v}_j$'s is a linear combo of the others.

A collection of vectors in $\mathbb{R}^n$ which <u>span $\mathbb{R}^n$</u> <u>and</u> are <u>linearly independent</u> is called a <u>basis</u> for $\mathbb{R}^n$.

to be continued...