Math 2250-3
Friday 10/29

See 2nd computer project in these notes
Due Nov. 10 (Wed)
you *may* use Matlab, but support is in Maple.

We're going through the algorithm to solve the constant coefficient linear homogeneous DE

$$\mathcal{L}(y) = 0$$

where $\mathcal{L}(y) = y^{(n)} + a_{n-1}y^{(n-1)} + \dots + a_2 y'' + a_1 y' + a_0 y$. $\qquad a_j$'s constant.

In all cases, trying $y = e^{rx}$ leads to

$$\mathcal{L}(y) = (\underbrace{r^n + a_{n-1}r^{n-1} + \dots + a_1 r + a_0}_{p(r)})e^{rx}$$

so $\mathcal{L}(y) = 0$ exactly when r is a root of the charact poly $p(r)$.

We've understood
Case I : distinct real roots
Case II : repeated roots
and are discussing
Case III : complex roots.

$$\boxed{\text{Euler: } e^{iy} = \cos y + i \sin y}$$

so, for $z = x + iy$, define

$$\boxed{e^z := e^x e^{iy} = e^x(\cos y + i\sin y)}$$

<u>Def</u> : If $f(x) + ig(x)$ is a complex-valued function of the real variable x,

then $(f(x) + ig(x))' := f'(x) + ig'(x)$.

So if $\mathcal{L}$ is a linear differential operator with real coefficients, we can talk about complex solutions to

$$\mathcal{L}(y) = 0.$$

If $y = f + ig$, then $\mathcal{L}(y) = \mathcal{L}(f) + i\mathcal{L}(g)$, so $\mathcal{L}(y) = 0$ iff $\mathcal{L}(f) = 0$ and $\mathcal{L}(g) = 0$.

<u>Theorem</u> : If r is a <u>complex</u> root of the characteristic polynomial, then $y = e^{rx}$ is a complex sol'n of $\mathcal{L}(y) = 0$.

<u>why</u> All we used to connect roots to solutions was $\boxed{\frac{d}{dx}e^{rx} = re^{rx}}$

we check this on the next page.

Let $r = a+bi$

then $$e^{rx} = e^{(a+bi)x} = e^{ax}(\cos bx + i\sin bx)$$

so $$\frac{d}{dx}(e^{rx}) = \frac{d}{dx}(\%) = e^{ax}\left[a(\cos bx + i\sin bx) + -b\sin bx + ib\cos bx\right]$$
$$= e^{ax}\left[a\cos bx - b\sin bx + i(a\sin bx + b\cos bx)\right]$$
$$\stackrel{?}{=} (a+bi)\,e^{ax}(\cos bx + i\sin bx) \quad \checkmark$$
$$= re^{rx}$$

∎

Now, let $r = a \pm ib$ be conjugate roots of the ~~charactor~~ characteristic poly $p(r)$

$$e^{(a+ib)x} = e^{ax}(\cos bx + i\sin bx)$$
$$e^{(a-ib)x} = e^{ax}(\cos(-bx) + i\sin(-bx))$$
$$= e^{ax}(\cos bx - i\sin bx)$$ since cos is even and sine is odd fcn.

From theorem, and since $\mathcal{L}$ is linear, get complex sol'ns

$$y = C_1 e^{(a+ib)x} + C_2 e^{(a-ib)x}$$
$$= C_1 e^{ax}(\cos bx + i\sin bx) + C_2 e^{ax}(\cos bx - i\sin bx)$$
$$= e^{ax}\cos bx\,(C_1 + C_2) + e^{ax}\sin bx\,(iC_1 - iC_2)$$
$$= c_1 \underbrace{e^{ax}\cos bx}_{y_1} + c_2 \underbrace{e^{ax}\sin bx}_{y_2}$$

$\rightarrow$ gives 2 <u>real</u> linearly ind sol'ns

Case III complex roots

Each pair of conjugate roots $a+bi, a-bi$ yield a pair of linearly ind. sol'ns
$$e^{ax}\cos bx,\ e^{ax}\sin bx$$

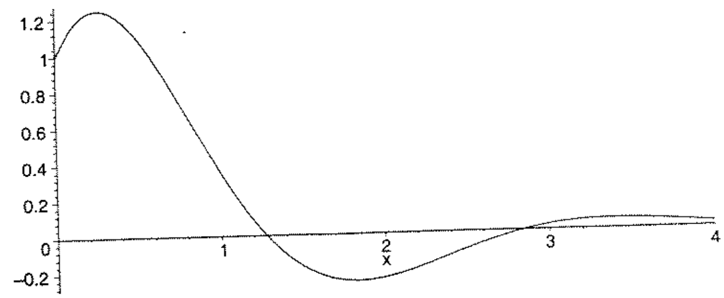
If $r = a \pm bi$ appears with multiplicity >1 in the characteristic poly, you also get sol'ns
$$xe^{ax}\cos bx,\ xe^{ax}\sin bx$$
$$\vdots$$

following the same algorithm as for repeated real roots as in Case II.

example 1

$$\begin{cases} y'' + 2y' + 5 = 0 \\ y(0) = 1 \\ y'(0) = 2 \end{cases}$$

```
> with(plots):
Warning, the name changecoords has been redefined
> plot(exp(-x)*(cos(2*x)+1.5*sin(2*x)),x=0..4,color=black);
```



$$p(r) = r^2 + 2r + 5 = 0$$

$$r = \frac{-2 \pm \sqrt{4-20}}{2} = \frac{-2 \pm 4i}{2} = -1 \pm 2i$$

$$e^{(-1+2i)x} = e^{-x}(\cos 2x + i \sin 2x)$$

$$y(x) = c_1 e^{-x} \cos 2x + c_2 e^{-x} \sin 2x$$

$$y(0) = 1 = c_1$$

$$y'(0) = 2 = -c_1 + 2c_2 \leftarrow \text{why?}$$

$$2 = -1 + 2c_2$$

$$3/2 = c_2$$

$$\boxed{y(x) = e^{-x}\left(\cos 2x + 3/2 \sin 2x\right)}$$

§5.4 Mechanical vibrations (= application, so $x(t)$ instead of $y(x)$).

$$\underbrace{m}_{\text{mass}} x'' + \underbrace{c}_{\text{coeff of friction}} x' + \underbrace{k}_{\text{Hooke's constant.}} x = 0$$

(see lecture notes 10/22 page 3 for derivation)

- interpret example 1 in this context.

We study this extremely important DE in stages:

Case 1 Free undamped motion

Free ↑ no driving force; undamped ↑ $c = 0$

$$mx'' + kx = 0$$

$$x'' + \frac{k}{m} x = 0$$

$$x'' + \omega_0^2 x = 0 \qquad \left(\omega_0 := \sqrt{\tfrac{k}{m}}\right)$$

$$\boxed{\begin{array}{l} x(t) = A \cos \omega_0 t + B \sin \omega_0 t \\ \text{simple harmonic motion} \end{array}}$$

if $x = e^{rt}$, $p(r) = r^2 + \omega_0^2 = 0$

$$r = \pm i\omega_0$$

$$e^{i\omega_0 t} = \cos \omega_0 t + i \sin \omega_0 t$$

The geometry of simple harmonic motion

$$x(t) = A\cos\omega_0 t + B\sin\omega_0 t = C\cos(\omega_0 t - \alpha)$$

ω_0 = angular frequency (units = radians/sec, if t is meas. in sec).

ν = frequency = $\frac{\omega_0}{2\pi}$ = cycles/sec.

$T = \frac{2\pi}{\omega_0}$ = period = secs/cycle

C = amplitude : notice $-C \le x(t) \le C$.

α = phase angle.

A, B are related to C, α via trig identities.

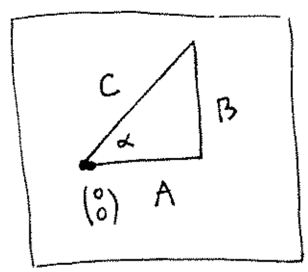
$$C\cos(\omega_0 t - \alpha) = C\left(\cos\omega_0 t\cos(-\alpha) - \sin\omega_0 t\sin(-\alpha)\right)$$
$$= \underbrace{(C\cos\alpha)}_{A}\cos\omega_0 t + \underbrace{(C\sin\alpha)}_{B}\sin\omega_0 t$$

so $A^2 + B^2 = C^2 \longrightarrow C = \sqrt{A^2+B^2}$

$\frac{A}{C} = \cos\alpha \longrightarrow \cos\alpha = \frac{A}{\sqrt{A^2+B^2}}$

$\frac{B}{C} = \sin\alpha \longrightarrow \sin\alpha = \frac{B}{\sqrt{A^2+B^2}}$

} so can find α with inverse trig fcns



A, B need not both be positive – in which case α will not give a triangle in 1st quadrant.

example : $\boxed{x(t) = \frac{1}{2}\cos 10t - \sin 10t}$

solves IVP

$$\begin{cases} x'' + 100x = 0 \\ x(0) = .5 \\ x'(0) = -10 \end{cases}$$

(see p. 316).

$A = .5$
$B = -1$

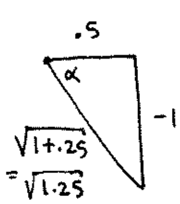
$\omega_0 = 10$

$C = \sqrt{1.25} \approx 1.12$

$\alpha = \sin^{-1}\left(\frac{-1}{\sqrt{1.25}}\right) = \tan^{-1}\left(\frac{-1}{.5}\right)$

because in quad IV don't use $\cos^{-1}$.

$\alpha \cong -1.107$ radians



$$\boxed{x(t) \cong 1.12\cos\left(10t + 1.107\right)}$$

```
> plot(.5*cos(10*t)-sin(10*t),t=-.2..1,color=black);
```