Math 2250-3
Wed 27 Oct

HW for Wed 11/3

5.3 3, 10, 14, 21, 22, 24, 29, 33, 37 (1)
5.4 4, 5, 7, 10, 12, 15, 17, 18, 23 — just find sol'ns $x(t)$, in appropriate form.
5.5 3, 4, 12, 13, 19, 34, 37, 43, 49, 50, 52

$$\mathcal{L}(y) = y^{(n)} + p_{n-1}y^{(n-1)} + \cdots + p_2 y'' + p_1 y' + p_0 y, \qquad p_j(x), f \text{ continuous on } I$$

$\mathcal{L}(y) = 0$ homogeneous

$\mathcal{L}(y) = f$ inhomogeneous.

We know

- IVP $\begin{cases} \mathcal{L}(y) = f \\ y(a) = b_0 \\ y'(a) = b_1 \\ \vdots \\ y^{(n-1)}(a) = b_{n-1} \end{cases}$ has unique solutions

- Solution space to $\mathcal{L}(y) = 0$ is n-dimensional
- You can use the Wronskian to see if you've found a basis (how?)

And ("new" today)

- The general solution to $\mathcal{L}(y) = f$ is $y = y_p + y_H$ (y_p: partic soltn; y_H: general soltn to homog. eqtn.)

Because

(i) If $\mathcal{L}(y_p) = f$ and $\mathcal{L}(y_H) = 0$

then $\mathcal{L}(y_p + y_H) = \mathcal{L}(y_p) + \mathcal{L}(y_H) = f + 0 = f$

so $y_p + y_H$ is always a soltn.

(ii) If $\mathcal{L}(y_p) = f$ and also $\mathcal{L}(y) = f$

then $\mathcal{L}(y - y_p) = f - f = 0$

so $y - y_p = y_h$ (a homog sol'n)

$y = y_p + y_h$.

<u>Example 4</u> On Monday we showed that for

$$\mathcal{L}(y) = y''' - 3y'' - 4y' + 12y$$

the soltns to $\mathcal{L}(y) = 0$ are

$$y_H(x) = c_1 e^{2x} + c_2 e^{-2x} + c_3 e^{3x} \qquad (\{e^{2x}, e^{-2x}, e^{3x}\} \text{ are a basis for the sol'n space})$$

y_H: book writes $y_c(x)$, for "complementary sol'n".

Find the full solution to

$$y''' - 3y'' - 4y' + 12y = 6$$

hint: try a particular sol'n which is constant.

Example 5 Rework the chapter 1 problem

$$y' - 7y = 14$$

How to solve constant coefficient linear homogeneous DE's

$$\mathcal{L}(y) = y^{(n)} + a_{n-1}y^{(n-1)} + \dots + a_1 y' + a_0 y = 0 \qquad a_j\text{'s constant}$$

step 1 try $y = e^{rx}$.

Then $\mathcal{L}(y) = (\underbrace{r^n + a_{n-1}r^{n-1} + \dots + a_1 r + a_0}_{p(r),\ \text{the characteristic polynomial}})e^{rx}$

So any root of $p(r)$ yields a solution

Now there a several cases of increasing complexity.

Case I If $p(r)$ has n distinct (different) real roots $r_1, r_2, \dots r_n$

then $y_H(x) = c_1 e^{r_1 x} + c_2 e^{r_2 x} + \dots + c_n e^{r_n x}$ is the general soltn (i.e. $\{e^{r_1 x}, e^{r_2 x}, \dots e^{r_n x}\}$ is a basis)

(Example 4 Monday is this case)

Case II Repeated roots (real).

(recall Monday example)

If $(r-\alpha)^k$ is a root of $p(r)$ then $e^{\alpha x}, xe^{\alpha x}, \dots x^{k-1}e^{\alpha x}$ are k linearly ind. sol'ns [see text for proofs].

So, if $p(r) = (r-r_1)^{k_1} \cdots (r-r_j)^{k_j}$

Then

$$y_H(x) = c_1 e^{r_1 x} + c_2 x e^{r_1 x} + \dots + c_n x^{k_j - 1} e^{r_j x}$$

Example 6 $\quad y'''' - y''' = 0$ $\quad\leftarrow$ you could also do this by antidifferentiation.

$$p(r) = r^4 - r^3 = r^3(r-1)$$

so $$\boxed{y_H(x) = \underset{\substack{\uparrow \\ 1 = e^{0x}}}{c_1} + c_2 x + c_3 x^2 + c_4 e^x}$$

Case III $\quad$ Complex roots: exponential fcns still work, you just need to remember Euler's formula.

Recall from Taylor series in Calc.

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots \qquad = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

If we write $i = \sqrt{-1}$ then

$$e^{ix} = 1 + ix + \frac{(ix)^2}{2!} + \frac{(ix)^3}{3!} + \ldots$$

$$= 1 + ix - \frac{x^2}{2!} - \frac{ix^3}{3!} + \frac{x^4}{4!} + \frac{ix^5}{5!} - \frac{x^6}{6!} - \frac{ix^7}{7!} + \ldots$$

$\cos x$!

$i \sin x$

$$\boxed{e^{ix} = \cos x + i \sin x} \qquad \text{Euler's formula}$$

Define, for $z = x + iy$

$$\boxed{e^z = e^{x+iy} := e^x e^{iy} = e^x(\cos y + i \sin y)}$$

Now, return to the DE $L(y) = 0$, with $r = a \pm ib$ roots of the characteristic polynomial. We claim

$$e^{(a+ib)x}, \quad e^{(a-ib)x} \quad \text{are (complex) solutions.}$$

$\uparrow$ $(f(x) + ig(x))' := f'(x) + ig'(x)$

$L(f + ig) = L(f) + iL(g)$ so if $y = f + ig$ is a complex solution, this is equivalent to $L(f) = 0 = L(g)$

This follows from the fact that

$$\boxed{\frac{d}{dx} e^{(a+ib)x} = (a+ib) e^{(a+ib)x}}$$

$\uparrow$

a good exercise in trig identities, p. 315

Thus if $r = a \pm ib$ are conjugate roots of the characteristic poly $p(r) = 0$
get sol'ns

$$C_1 e^{(a+ib)x} + C_2 e^{(a-ib)x}$$
$$= C_1 e^{ax}(\cos bx + i\sin bx) + C_2 e^{ax}(\cos bx - i\sin bx)$$
$$= c_1 e^{ax}\cos bx + c_2 e^{ax}\sin bx$$

$\left(\begin{array}{l}\cos(-bx) = \cos bx \\ \sin(-bx) = -\sin bx\end{array}\right)$

linearly ind. real sol'ns!

<u>example</u> $y'' + 4y = 0$

$p(r) = r^2 + 4 = 0 \qquad r = \pm 2i$

$e^{\pm 2ix} = \cos 2x \pm i\sin 2x$

$y_H(x) = c_1 \cos 2x + c_2 \sin 2x.$