Math 2250-3
Monday 10/25
§5.2 → begin with page 5 Friday.

Example 1: Find all solutions to the 3rd order linear homogeneous differential equation

$$\mathcal{L}(y) = y'''(x) - 3y''(x) - 4y' + 12y = 0$$

let's try $y = e^{rx}$:
$y' = re^{rx}$
$y'' = r^2 e^{rx}$
$y''' = r^3 e^{rx}$

$$\mathcal{L}(y) = \underbrace{(r^3 - 3r^2 - 4r + 12)}_{\text{want roots of the characteristic polynomial}} e^{rx}$$

possible integer roots divide 12
$\pm 1, \pm 2, \pm 3, \pm 4, \pm 6, \pm 12,$
$\times$ ↑

$r = 2$: $8 - 12 - 8 + 12 = 0$ ✓

$$\begin{array}{r|l} & r^2 - r - 6 \\ \hline r-2 & r^3 - 3r^2 - 4r + 12 \\ & r^3 - 2r^2 \\ \hline & -r^2 - 4r \\ & -r^2 + 2r \\ \hline & -6r + 12 \\ & -6r + 12 \\ \hline & 0 \end{array}$$

$p(r) = (r-2)(r^2 - r - 6) = (r-2)(r+2)(r-3)$.

deduce
$y_1 = e^{2x}$
$y_2 = e^{-2x}$
$y_3 = e^{3x}$
are solutions.

$\mathcal{L}$ is a linear operator, (see page 3)

so
$$\mathcal{L}(c_1 y_1 + c_2 y_2 + c_3 y_3) = \mathcal{L}(c_1 y_1) + \mathcal{L}(c_2 y_2) + \mathcal{L}(c_3 y_3)$$
$$= c_1 \mathcal{L}(y_1) + c_2 \mathcal{L}(y_2) + c_3 \mathcal{L}(y_3)$$
$$= 0$$

So $c_1 y_1 + c_2 y_2 + c_3 y_3$ also satisfies this linear homogeneous DE.

In physics you call this the principle of superposition.

[If $\mathcal{L}(y) = f$, $\mathcal{L}(z) = g$ then $\mathcal{L}(y+z) = f + g$].
$\mathcal{L}(ay + bz) = af + bg$
a, b const.

So if V is our solution space to $\mathcal{L}(y) = 0$, we see that any
$$c_1 y_1 + c_2 y_2 + c_3 y_3 \in V.$$

Is $\text{span}\{y_1, y_2, y_3\}$ all of V?
Are y_1, y_2, y_3 linearly independent?

(continue on next page!)

Example 2 : Solve the initial value problem:

$$\text{IVP}\begin{cases} y'''(x) - 3y''(x) - 4y' + 12y = 0 \\ y(0) = 0 \\ y'(0) = -3 \\ y''(0) = 5 \end{cases}$$

$y_1 = e^{2x}$
$y_2 = e^{-2x}$
$y_3 = e^{3x}$

$$y(x) = c_1 y_1 + c_2 y_2 + c_3 y_3$$
$$y' = c_1 y_1' + c_2 y_2' + c_3 y_3'$$
$$y'' = c_1 y_1'' + c_2 y_2'' + c_3 y_3''$$

$$\begin{bmatrix} y \\ y' \\ y'' \end{bmatrix} = \begin{bmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$$

If you have n solutions to a linear homogeneous DE, the matrix for which column j is $\begin{bmatrix} y_j \\ y_j' \\ \vdots \\ y_j^{n-1} \end{bmatrix}$ is called the Wronskian matrix. Its determinant is called the Wronskian $W(y_1, y_2, y_n)$

$$\begin{bmatrix} y \\ y' \\ y'' \end{bmatrix} = \begin{bmatrix} e^{2x} & e^{-2x} & e^{3x} \\ 2e^{2x} & -2e^{-2x} & 3e^{3x} \\ 4e^{2x} & 4e^{-2x} & 9e^{3x} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$$

at $x=0$ this is the matrix eqtn.

$$\begin{bmatrix} 0 \\ -3 \\ 5 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 2 & -2 & 3 \\ 4 & 4 & 9 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$$

Could we have solved any IVP for this DE, at $x=0$? Could the Wronskian explain why?

```
> A:=matrix(3,4,[1,1,1,0,
                 2,-2,3,-3,
                 4,4,9,5]);
```

$$A := \begin{bmatrix} 1 & 1 & 1 & 0 \\ 2 & -2 & 3 & -3 \\ 4 & 4 & 9 & 5 \end{bmatrix}$$

```
> rref(A);
```

$$\begin{bmatrix} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

$c_1 = -2$
$c_2 = 1$
$c_3 = 1$

```
>
```

$$\boxed{y(x) = -2e^{2x} + e^{-2x} + e^{3x}}$$

Since $W(y_1, y_2, y_3) \neq 0$, every IVP (at $x=0$) solution is a unique linear combination of y_1, y_2, y_3.

See theory pages. theorems 2 & 3. $\rightarrow$ In particular, they span the solution space and are linearly independent

So they are a basis

Theory pages

an n^{th} order differential operator

$$\mathcal{L}(y) = P_n(x)\, y^{(n)} + P_{n-1}(x)\, y^{(n-1)} + \dots + P_1(x)\, y'(x) + P_0(x)\, y(x)$$

is called linear, because

$$* \begin{cases} \mathcal{L}(y_1 + y_2) = \mathcal{L}(y_1) + \mathcal{L}(y_2) \\ \mathcal{L}(cy_1) = c\mathcal{L}(y_1) \end{cases}$$

The differential equation

$$\mathcal{L}(y) = F(x)$$

is homogeneous if $F(x) \equiv 0$. Otherwise it is inhomogeneous

Theorem 1 : The solution space to the linear, homogeneous, DE

$$\mathcal{L}(y) = 0$$

is a vector space, i.e. it is closed under addition and scalar multiplication.
Thus if $y_1, \dots, y_n$ are solutions, so is

$$c_1 y_1 + c_2 y_2 + \dots + c_n y_n .$$

why: If y_1, y_2 are solutions, then $\mathcal{L}(y_1 + y_2) \overset{\text{by } *}{=} \mathcal{L}(y_1) + \mathcal{L}(y_2) = 0 + 0$
so $y_1 + y_2$ is.

If y_1 is a solution, then
$\mathcal{L}(cy_1) = c\mathcal{L}(y_1) = 0$, so cy_1 is too.

More generally,

$$\begin{aligned} \mathcal{L}(c_1 y_1 + c_2 y_2 + \dots + c_n y_n) &= \mathcal{L}(c_1 y_1) + \mathcal{L}(c_2 y_2 + \dots + c_n y_n) \\ &= c_1 \underbrace{\mathcal{L}(y_1)}_{0} + \mathcal{L}(\underbrace{c_2 y_2}_{0}) + \mathcal{L}(c_3 y_3 + \dots + c_n y_n) \\ &= \dots = 0 + 0 + \dots + 0 = 0. \end{aligned}$$

Theorem 2 : Let

$$\text{IVP} \begin{cases} \mathcal{L}(y) = y^{(n)} + p_{n-1}(x)\, y^{(n-1)} + \dots + p_0(x)\, y \equiv 0 \\ y(a) = b_0 \\ y'(a) = b_1 \\ \vdots \\ y^{(n-1)}(a) = b_{n-1} \end{cases}$$

If $p_{n-1}, \dots, p_0$ are continuous on an interval I and if $a \in I$.
Then there is always exactly 1 solution to this initial value problem.

why: makes intuitive sense.
more explanation later in course.

Theorem 3: Let $\mathcal{L}(y) = y^{(n)} + p_{n-1}y^{(n-1)} + \dots + p_1 y' + p_0 y$ be as in theorem 2. Then the solution space to $\mathcal{L}(y) = 0$ has dimension n. Thus if we can find n linearly independent solutions they will span the solution space.

why: Using the existence-uniqueness result, Theorem 2, we can find (well, actually, there exist) $y_1 \dots y_n$ solving $\mathcal{L}(y_j) = 0$, with initial values

$$\begin{bmatrix} y_1(a) \\ y_1'(a) \\ \vdots \\ y^{(n-1)}(a) \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad \begin{bmatrix} y_2(a) \\ y_2'(a) \\ \vdots \\ y_2^{(n-1)}(a) \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \dots \begin{bmatrix} y_n(a) \\ y_n'(a) \\ \vdots \\ y_n^{(n-1)}(a) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}.$$

$\{y_1, \dots y_n\}$ are a basis:

(a) span: Let $y(x)$ solve $\mathcal{L}(y) = 0$. Then

$y(a) = b_0$
$y'(a) = b_1$
$\vdots$
$y^{(n-1)} = b_{n-1}$

some #'s $b_0, - b_{n-1}$

compare $y(x)$ to $b_0 y_1(x) + b_1 y_2(x) + \dots + b_{n-1} y_{n-1}(x)$
these 2 solutions solve the same IVP, so they must be the same function, by Theorem 2.
i.e. $y(x) = b_0 Y_1 + b_1 Y_1 + \dots + b_{n-1} Y_{n-1}$

(b) linearly ind: if
$$c_1 y_1 + c_2 y_2 + \dots + c_n y_n = 0$$
then
$$c_1 y_1' + c_2 y_2' + \dots + c_n y_n' = 0$$
$$\vdots$$
$$c_1 y_n^{(n-1)} + \dots + c_n y_n^{(n-1)} = 0$$

at $x = a$ deduce
$c_1 = 0$
$c_2 = 0$
$\vdots$
$c_n = 0$

$\blacksquare$

Theorem 4: Let $\mathcal{L}(y)$ be as in Theorems 2 & 3.
Let $\{y_1, \dots y_n\}$ be n solutions to $\mathcal{L}(y) = 0$. on the interval I.
Let
$$W = \det \begin{bmatrix} y_1 & y_2 & \dots & y_n \\ y_1' & y_2' & & y_n' \\ \vdots & \vdots & & \vdots \\ y_1^{(n-1)} & y_2^{(n-1)} & & y_n^{(n-1)} \end{bmatrix}$$
(the Wronskian).

Let $a \in I$. Then if $W(a) \neq 0$, $y_1 \dots y_n$ are a basis!
(so you don't need to compute W at every x!)

why: $W(a) \neq 0$ means every IVP at a has a unique solution expressed as a linear combination of $y_1, y_2 \dots y_n$.

Example 3   Find the solution space to

$$y'' - 6y' + 9 = 0$$

try $y = e^{rx}$
$y' = re^{rx}$
$y'' = r^2 e^{rx}$

$$\mathcal{L}(y) = (r^2 - 6r + 9)e^{rx} = 0$$

$$(r-3)^2 = 0$$

$$y_1 = e^{3x}.$$

Where's y_2?

trick: try $y_2 = xe^{3x}$
$y_2' = (1+3x)e^{3x}$ (product rule)
$y_2'' = (3 + 3(1+3x))e^{3x}$

$$\mathcal{L}(y_2) = e^{3x}\begin{bmatrix} x(9-18+9) \\ +1(6-6) \end{bmatrix} = 0!$$

$$\begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix} = \begin{bmatrix} e^{3x} & xe^{3x} \\ 3e^{3x} & e^{3x}(1+3x) \end{bmatrix}$$

$$W(0) = \det\begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix} = 1$$

so $\{y_1, y_2\}$ is a basis (see Theorem 4 p 4)

$$\boxed{y(x) = c_1 e^{3x} + c_2 xe^{3x}}$$

(note, $W(x) = e^{9x}$ is never 0).

example 3 is typical of double roots. if $p(r) = (r-\alpha)^2 = r^2 - 2\alpha + \alpha^2$,
for the 2^{nd} order, linear, homogeneous, const-coeff DE

$$y'' - 2\alpha y' + \alpha^2 y = 0$$

then $y_1 = e^{\alpha x}$, $y_2 = xe^{\alpha x}$ are a basis for the solution space

note
$$\left.\begin{aligned} y_2 &= xe^{\alpha x} \\ y_2' &= e^{\alpha x}(\alpha x + 1) \\ y_2'' &= e^{\alpha x}(\alpha + \alpha(\alpha x + 1)) \end{aligned}\right]\begin{aligned} &\cdot \alpha^2 \\ &\cdot -2\alpha \\ &\cdot 1 \end{aligned}$$

$$\mathcal{L}(y_2) = e^{\alpha x}\begin{bmatrix} x(\alpha^2 - 2\alpha^2 + \alpha^2) \\ +1(-2\alpha + 2\alpha) \end{bmatrix} = 0.$$