Math 2250-3
Wed 10/20

HW for Wed 10/27.

4.7	1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 13, 15, 17, 21, 23, 25
5.1	2, 9, 11, 17, 27, 29, 30, 31, 34, 35, 39
5.2	2, 5, 9, 11, 13, 21, 22, 25, 26

§4.7 General Vector spaces
(= §4.5 in old edition)

Still need to prove a neat fact,
which was in notes last Friday, Oct 15. p.3:

If V is n-dimensional, then

(a) If $\{\vec{v}_1, \ldots \vec{v}_n\}$ span V they are <u>also</u> independent → i.e. basis

(b) If $\{\vec{v}_1, - \vec{v}_n\}$ are indep., they <u>also</u> span → i.e. basis

<u>pf</u>: (a) If $\{\vec{v}_1, - \vec{v}_n\}$ is <u>dep</u>,

then some $\vec{v}_j$ is a linear combo of others, call it $\vec{v}_n$.

i.e. $\vec{v}_n = c_1\vec{v}_1 + \cdots + c_{n-1}\vec{v}_{n-1}$

but then $\text{span}\{\vec{v}_1, - \vec{v}_n\} = \text{span}\{\vec{v}_1, \ldots \vec{v}_{n-1}\}$

since $d_1\vec{v}_1 + \cdots + d_n\vec{v}_n = d_1\vec{v}_1 + \cdots + d_{n-1}\vec{v}_{n-1} + d_n(c_1\vec{v}_1 + \cdots + c_{n-1}\vec{v}_{n-1})$!

so would have $(n-1)$ vectors spanning n-dim'l space — can't happen. ∎

(b) If $\{\vec{v}_1, \ldots \vec{v}_n\}$ don't span, let $\vec{w}$ be a vector in V which is not a linear combo of $\vec{v}_1, - \vec{v}_n$.

Then $\{\vec{v}_1, \ldots \vec{v}_n, \vec{w}\}$ are independent since

$$c_1\vec{v}_1 + \cdots + c_n\vec{v}_n + d\vec{w} = \vec{0} \Rightarrow d = 0 \quad (\text{else } \vec{w} \text{ } \underline{\text{is}} \text{ in span of } \vec{v}_j\text{'s})$$

$$\Rightarrow c_1 = \cdots = c_n = 0 \quad ∎$$

<u>example</u>: We already seen all this for $\mathbb{R}^n$.

<u>example</u> $P^2 = \text{span}\{1, x, x^2\} \subset \mathcal{F}$ (P^2 = polys in x of degree ≤ 2).

$= \{a_0 + a_1 x + a_2 x^2 \text{ s.t. } a_i \in \mathbb{R}\}$

$\vec{v}_1 = 1$
$\vec{v}_2 = x$
$\vec{v}_3 = x^2$

these vectors span P^2, by definition.
are they a basis?

$$c_1 1 + c_2 x + c_3 x^2 \equiv 0$$
$$\Rightarrow c_2 + 2c_3 x \equiv 0 \quad (\text{take deriv})$$
$$\Rightarrow 2c_3 \equiv 0 \Rightarrow c_3 = 0 \Rightarrow c_2 = 0 \Rightarrow c_1 = 0$$

So P^2 is 3-dimensional.

example continued:

Why can we write

$$\frac{6x}{(x-1)(x+1)(x+2)} = \frac{A}{x-1} + \frac{B}{x+1} + \frac{C}{x+2}$$ for some unique choice of A, B, C?

$$= \frac{A(x+1)(x-2) + B(x-1)(x-2) + C(x-1)(x+1)}{(x-1)(x+1)(x+2)}$$

$$\boxed{6x = A(x+1)(x+2) + B(x-1)(x+2) + C(x-1)(x+1)}$$

$\vec{v}_1 = (x+1)(x-2)$
$\vec{v}_2 = (x-1)(x-2)$
$\vec{v}_3 = (x-1)(x+1)$

Are $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ a basis for P_2?
(If so, A, B, C exist and are unique)

Well, are $\vec{v}_1, \vec{v}_2, \vec{v}_3$ linearly independent?

$$c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 = \vec{0}$$

$$c_1(x+1)(x+2) + c_2(x-1)(x+2) + c_3(x-1)(x+1) \equiv 0$$

$$\left.\begin{aligned} x=1 &\Rightarrow -2c_1 = 0 \\ x=2 &\Rightarrow 3c_3 = 0 \\ x=-1 &\Rightarrow 6c_2 = 0 \end{aligned}\right\} \text{yes!}$$

so by logic of previous page we have a basis!

$$\boxed{\text{linear algebra explains why partial fractions works!}}$$

Of course, you already believed it worked.
And you've used 2 approaches to do partial fractions:

(1) plug in into boxed numerator equality:

$x = 1 \Rightarrow 6 = 6A \qquad A = 1$
$x = -2 \Rightarrow -12 = 3C \qquad C = -4$
$x = -1 \Rightarrow -6 = -2B \qquad B = 3$

$$\frac{6x}{(x-1)(x+1)(x+2)} = \frac{1}{x-1} + \frac{3}{x+1} - \frac{4}{x+2}$$

(2) write boxed equality using the natural basis $\{1, x, x^2\}$:

$$6x = x^2(A+B+C) + x(3A+B) + 1(2A-2B-C)$$

because $\{1, x, x^2\}$ are lin ind.

$$\begin{array}{ll} x^2 & A+B+C = 0 \\ x & 3A+B = 6 \\ 1 & 2A-2B-C = 0 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 3 & 1 & 0 & 6 \\ 2 & -2 & -1 & 0 \end{array}\right]$$

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> A:=matrix(3,4,[1,1,1,0,
                 3,1,0,6,
                 2,-2,-1,0]);
>
```

$$A := \begin{bmatrix} 1 & 1 & 1 & 0 \\ 3 & 1 & 0 & 6 \\ 2 & -2 & -1 & 0 \end{bmatrix}$$

```
> rref(A);
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$$\checkmark \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & -4 \end{bmatrix}$$

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<u>example</u> Let $\mathcal{L}(y) = y'' - 4y'$

$\mathcal{L}$ is a <u>linear</u> operator on (twice differentiable) functions in $\mathcal{F}$,

i.e. $\mathcal{L}(y_1+y_2) = \mathcal{L}(y_1) + \mathcal{L}(y_2)$ $\quad \because \quad (y_1+y_2)'' - 4(y_1+y_2) = (y_1''-4y_1) + (y_2''-4y_2)$

$\mathcal{L}(cy_1) = c\mathcal{L}(y_1)$ $\quad \because \quad (cy_1)'' - 4(cy_1) = c(y_1''-4y_1)$

Thus, the solution space to the homogeneous differential equation

$$\mathcal{L}(y) = 0$$

is a subspace of $\mathcal{F}$.

(Since if $\mathcal{L}(y_1) = \mathcal{L}(y_2) = 0$ then $\mathcal{L}(y_1+y_2) = 0$
$\mathcal{L}(cy_1) = 0$.

Let's find the general sol'n to

$$y'' - 4y' = 0.$$

if $v = y'$ then $v' - 4v = 0$

$$v' = 4v$$
$$v(x) = \tilde{c}_1 e^{4x}$$
$$y' = \tilde{c}_1 e^{4x}$$
$$y = \frac{\tilde{c}_1}{4} e^{4x} + c_2$$
$$y = c_1 e^{4x} + c_2 \cdot 1$$

$\{e^{4x}, 1\}$ span W.

are they lin ind?

$$c_1 e^{4x} + c_2 \cdot 1 \equiv 0$$
$$\Rightarrow 4c_1 e^{4x} + 0 \equiv 0 \qquad \Rightarrow c_1 = 0$$
$$\Rightarrow c_2 = 0 \quad \checkmark$$

W is a 2-dim'l vector space.

<u>example cont</u>

solve $\begin{cases} y'' - 4y' = 0 \\ y(0) = 1 \\ y'(0) = 6 \end{cases}$

$$y(x) = c_1 e^{4x} + c_2$$
$$y'(x) = 4c_1 e^{4x}$$

$$y(0) = 1 = c_1 + c_2$$
$$y'(0) = 6 = 4c_1$$

$$\Rightarrow c_1 = 3/2$$
$$c_2 = -1/2$$

$$\boxed{y(x) = \frac{3}{2} e^{4x} - \frac{1}{2}}$$