Math 2250-3
Fri 10/15

How's your vocabulary?

linear combination of $\vec{v}_1, \ldots \vec{v}_k$

span $\{\vec{v}_1, \ldots \vec{v}_k\}$

$\{\vec{v}_1, \ldots \vec{v}_k\}$ linearly independent
linearly dependent

V a vector space

W a subspace [not just any subset!]

Def $\{\vec{v}_1, \ldots \vec{v}_k\}$ is a basis for V iff

(a) $V = \text{span}\{\vec{v}_1, \ldots \vec{v}_k\}$

(b) $\{\vec{v}_1, \ldots \vec{v}_k\}$ linearly independent

this is equivalent to saying that each $\vec{v} \in V$ can be uniquely expressed as
$$\vec{v} = c_1\vec{v}_1 + c_2\vec{v}_2 + \ldots + c_k\vec{v}_k$$
in this case the linear combo coefficients are called the coords of $\vec{v}$ with respect to the basis $\{\vec{v}_1, \ldots \vec{v}_k\}$.

Example The standard basis $\{\vec{e}_1, \ldots \vec{e}_n\}$ of $\mathbb{R}^n$

where $\vec{e}_j = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ 0 \\ 0 \end{bmatrix} \leftarrow$ entry j

(a) span: $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = x_1\vec{e}_1 + x_2\vec{e}_2 + \ldots + x_n\vec{e}_n$

(b) linearly independent: if $c_1\vec{e}_1 + c_2\vec{e}_2 + \ldots + c_n\vec{e}_n = \vec{0}$

then $\begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$ so all c_j's $= 0$

Example (Wed page 1).

the plane $x + 2y - 3z = 0$ has basis $\left\{ \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} \right\}$

since $1 \quad 2 \quad -3 \mid 0$

(a) span: $z = t$, $y = s$, $x = 3t - 2s$ $\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = t\begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix} + s\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$

(b) if
$$c_1\begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix} + c_2\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} 3c_1 - 2c_2 \\ c_2 \\ c_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$
so $c_1 = c_2 = 0$.

Recall, we figured out that

(a) more than n vectors in $\mathbb{R}^n$ are always linearly dependent

(b) less than n vectors in $\mathbb{R}^n$ cannot span $\mathbb{R}^n$

- so each basis of $\mathbb{R}^n$ has exactly n vectors

  <u>and</u> $\{\vec{v}_1, \ldots \vec{v}_n\}$ is a basis for $\mathbb{R}^n$ iff

$$A = \left[\vec{v}_1 \middle| \vec{v}_2 \middle| \cdots \vec{v}_n\right] \qquad \text{satisfies } \operatorname{rref}(A) = I$$

<u>Def</u> For any vectorspace V, the dimension of V ($\dim(V)$) is defined ~~the~~ to be the number of vectors in every basis of V.

This definition only makes sense because

<u>Theorem</u> Every basis of V has the same number of vectors.

<u>pf</u>: (I will only worry about finite dim'l vector spaces).

Let $\{\vec{v}_1, \ldots \vec{v}_k\}$ $\quad \{\vec{w}_1, \ldots \vec{w}_{k+\ell}\}$ two basis candidates ($\ell > 0$)

suppose $\{\vec{v}_1, \ldots v_k\}$ span V.

Let's show $\{\vec{w}_1, \ldots \vec{w}_{k+\ell}\}$ are dependent:

we search for lin combo coef's s.t.

$$c_1\vec{w}_1 + c_2\vec{w}_2 + \cdots + c_{k+\ell}\vec{w}_{k+\ell} = 0.$$

Express each $\vec{w}_j$ as a linear combo of the v_i's:

$$c_1 \begin{bmatrix} a_{11}\vec{v}_1 \\ +a_{21}\vec{v}_2 \\ + \\ \vdots \\ +a_{k1}\vec{v}_k \end{bmatrix} + c_2 \begin{bmatrix} a_{12}\vec{v}_1 \\ +a_{22}\vec{v}_2 \\ + \\ \vdots \\ +a_{k2}\vec{v}_k \end{bmatrix} + \cdots + c_{k+\ell} \begin{bmatrix} a_{1\,k+\ell}\vec{v}_1 \\ +a_{2\,k+\ell}\vec{v}_2 \\ + \\ \vdots \\ +a_{k,k+\ell}\vec{v}_k \end{bmatrix} = \begin{bmatrix} 0\vec{v}_1 \\ +0\vec{v}_2 \\ + \\ \vdots \\ +0\vec{v}_k \end{bmatrix}$$

I succeed if I find $\vec{c} \neq \vec{0}$ s.t.

$$\begin{bmatrix} & A & \end{bmatrix} \begin{bmatrix} \\ \vec{c} \\ \\ \end{bmatrix} = \begin{bmatrix} \vec{0} \end{bmatrix}$$

But I can always do this because A has more columns than rows!

$$\boxed{\begin{array}{l} \text{Thus } \{\vec{v}_1, \ldots \vec{v}_k\} \text{ span} \Longrightarrow \{\vec{w}_1, \ldots \vec{w}_{k+\ell}\} \text{ dependent} \\ \text{logical equivalent: } \{\vec{w}_1, \ldots \vec{w}_{k+\ell}\} \text{ not dependent (i.e. independent)} \Longrightarrow \{\vec{v}_1, \ldots \vec{v}_k\} \text{ don't span!} \end{array}}$$

(p,q)	q T	q F
p T	(T,T)	~~(T,F)~~
p F	(F,T)	(F,F)

truth table for $p \Rightarrow q$ and $\text{not}\, q \Rightarrow \text{not}\, p$.

Thus, if $\{\vec{v}_1, \ldots \vec{v}_k\}$ is a basis, every other basis also has k vectors.

Useful:

Theorem If V is n-dimensional, then

(1) If $\vec{v}_1, \vec{v}_2, \ldots \vec{v}_n$ are linearly independent
they must also span V, and so are a basis

(2) If $\vec{v}_1, \ldots \vec{v}_n$ span V
they are automatically independent, and so are a basis.

example: This is consistent with our discussion of when n vectors in $\mathbb{R}^n$ are a basis — the conditions for linear ind. & spanning are the same, namely that

$$\text{rref}\left[\vec{v}_1 \middle| \vec{v}_2 \middle| \cdots \vec{v}_n\right] = I.$$

proof of thm:

(1): If $\vec{v}_1, \ldots \vec{v}_n$ are l.i. but don't span, then find $\vec{w}$ not in their span.
The larger set
$\{\vec{v}_1, \ldots \vec{v}_n, \vec{w}\}$ is still l.i., since if
$$c_1\vec{v}_1 + \cdots + c_n\vec{v}_n + d\vec{w} = 0$$

- $d=0 \Rightarrow$ all c_j's $=0$ by l.i. of v_j's.
- $d \neq 0$ can't be true since $\vec{w}$ not in span$\{\vec{v}_1 \ldots \vec{v}_n\}$.

but $(n+1)$ vectors in n-dim'l v-space cannot be independent (page 2)
so $\vec{w}$ didn't exist

(2) If $\vec{v}_1, \ldots \vec{v}_n$ span but are dependent, one of them (call it $\vec{v}_n$) is a linear combo of the others,

e.g. $\vec{v}_n = c_1\vec{v}_1 + \cdots + c_{n-1}\vec{v}_{n-1}$

then $d_1\vec{v}_1 + \cdots + d_{n-1}\vec{v}_{n-1} + d_n\underbrace{\vec{v}_n}_{= c_1\vec{v}_1 + \cdots + c_{n-1}\vec{v}_{n-1}}$ is in span$\{\vec{v}_1, \ldots \vec{v}_{n-1}\}$

So $\vec{v}_1, \ldots \vec{v}_{n-1}$ span V, which is not possible since $\dim V = n$.

Thus no $\vec{v}_j$ is a combo of the others,
so $\{\vec{v}_1, \ldots \vec{v}_n\}$ are independent

▨

Basis for Solution space

Our method for solving $A\vec{x} = \vec{0}$ always yields a basis for the solution space.

I will "prove" this by example:

Find a basis for the solution space to $A\vec{x} = \vec{0}$, for A given below:

```
> with(linalg):
Warning, the protected names norm and trace have been redefined and unprotected
> A:=matrix(4,6,[1,2,0,1,1,2,
                 2,4,1,4,1,7,
                -1,-2,1,1,-2,1,
                -2,-4,0,-2,-2,-4]);
```

$$A := \begin{bmatrix} 1 & 2 & 0 & 1 & 1 & 2 \\ 2 & 4 & 1 & 4 & 1 & 7 \\ -1 & -2 & 1 & 1 & -2 & 1 \\ -2 & -4 & 0 & -2 & -2 & -4 \end{bmatrix} \begin{matrix} 0 \\ 0 \\ 0 \\ 0 \end{matrix}$$

homog eqtn!

```
> rref(A);
```

$$\begin{array}{cccccc} & p & & r & t & s \end{array}$$
$$\begin{bmatrix} 1 & 2 & 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & 2 & -1 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{matrix} 0 \\ 0 \\ 0 \\ 0 \end{matrix}$$

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>
```

$x_6 = s$
$x_5 = t$
$x_4 = r$
$x_3 = -2r + t - 3s$
$x_2 = p$
$x_1 = -2p - r - t - 2s$

$$\vec{x} = p \underbrace{\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}}_{\vec{v}_1} + r \underbrace{\begin{bmatrix} -1 \\ 0 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}}_{\vec{v}_2} + t \underbrace{\begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}}_{\vec{v}_3} + s \underbrace{\begin{bmatrix} -2 \\ 0 \\ -3 \\ 0 \\ 0 \\ 1 \end{bmatrix}}_{\vec{v}_4}$$

(a) by construction, $\{\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4\}$ span the solution space

(b) suppose $c_1 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ 0 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + c_3 \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + c_4 \begin{bmatrix} -2 \\ 0 \\ -3 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

by focusing on entries 2, 4, 5, 6 (the cols of our free parameters)

we see
$c_1 = 0$
$c_2 = 0$
$c_3 = 0$
$c_4 = 0$

This shows linear independence!