Math 2250
Fri Oct 1.

Exam review session
(we'll go over practice exam, mainly.)
EMCB 112 tomorrow
(Sat Oct 2)
10:00 – 11:30 a.m.

Finish determinants,
and nifty formulas for A^{-1}
and Cramer's rule for solving $A\vec{x}=\vec{b}$.

Recall recursive def of det:

$$|A| = \sum_{j=1}^{n} a_{ij}C_{ij} \quad (\text{expansion accross row}_i(A))$$

$$= \sum_{i=1}^{n} a_{ij}C_{ij} \quad (\text{expansion down col}_j(A).$$

$$\overset{\substack{\left(\text{book writes } A_{ij}\right)\\ \downarrow}}{C_{ij}} = \underset{\substack{\uparrow\\ \text{precisely}\\ (-1)^{i+j}}}{\pm} M_{ij}$$

"Minor" precisely the $(n-1)\times(n-1)$ det of matrix obtained from A by deleting $\text{row}_i(A)$ & $\text{col}_j(A)$.

$$\begin{bmatrix} + & - & + & \cdots \\ - & + & & \\ + & & \ddots & \\ \vdots & & & \end{bmatrix}$$

- expanding down <u>any</u> column, or across <u>any</u> row yields same value, i.e. $|A|$.
- If A is (upper or lower) triangular, $|A|$ is the product of its diagonal terms.

Effects of elementary row operations (<u>or column ops</u>) on determinants:

(1) swapping two rows changes sign of determinant
→ checked this on Wednesday

(1b): So, if 2 rows are equal, $|A|=0$.
reason: Let $x=|A|$. If you swap the identical rows the new ~~at~~ matrix has det $= -x$.
But the new matrix is the old matrix, so $x=-x$, i.e. $2x=0$, $x=0$ ■

(2) multiplying a single row by a constant multiplies det by same const:

$$\begin{vmatrix} R_1 \\ R_2 \\ \vdots \\ cR_i \\ \vdots \\ R_n \end{vmatrix} \underset{\substack{\uparrow\\ \text{expand}\\ \text{across}\\ \text{row}_i}}{=} \sum_{j=1}^{n} (ca_{ij})\underset{\substack{\uparrow\\ \text{still } A\text{-cofactors!}}}{C_{ij}} = c\sum_{j=1}^{n} a_{ij}C_{ij} = c|A|$$

■

(the effect of this is that when we do row ops, if we factor a "c" out of a row we multiply it by the det of what's left to get original det.)

(3) replace $\text{row}_i(A)$ with $\text{row}_i(A) + c\,\text{row}_k(A)$:
Does not change det!!!

reason:

$$\text{row}_i \rightarrow \begin{vmatrix} R_1 \\ R_2 \\ \vdots \\ R_i + cR_k \\ \vdots \\ R_n \end{vmatrix} \underset{\substack{\uparrow \\ \text{expand} \\ \text{across row}_i}}{=} \sum_{j=1}^{n} (a_{ij} + ca_{kj}) \underset{\substack{\uparrow \\ \text{still A cofactors!}}}{C_{ij}} = \underbrace{\sum_{j=1}^{n} a_{ij} C_{ij}}_{\substack{\shortparallel \\ |A|}} + \underbrace{c\sum_{j=1}^{n} a_{kj} C_{ij}}_{\substack{\shortparallel \\ \begin{vmatrix} R_1 \\ R_2 \\ R_k \\ \vdots \\ R_k \\ \vdots \\ R_n \end{vmatrix} \begin{matrix} \\ \\ \leftarrow \text{row}_k \\ \\ \leftarrow \text{row}_i \\ \\ \\ \end{matrix} \nearrow 0}}$$

<u>example</u> (did on Wednesday)

$$\begin{vmatrix} 1 & 2 & -1 \\ 0 & 3 & 1 \\ 2 & -2 & 1 \end{vmatrix}$$

$$= \begin{array}{r} \\ \\ -2R_1+R_3 \end{array} \begin{vmatrix} 1 & 2 & -1 \\ 0 & 3 & 1 \\ 0 & -6 & 3 \end{vmatrix}$$

$$= \begin{array}{r} \\ \\ +2R_2+R_3 \end{array} \begin{vmatrix} 1 & 2 & -1 \\ 0 & 3 & 1 \\ 0 & 0 & 5 \end{vmatrix}$$

$$\begin{array}{l} \\ R_2/3 \\ R_3/5 \end{array} = 3 \cdot 5 \begin{vmatrix} 1 & 2 & -1 \\ 0 & 1 & 1/3 \\ 0 & 0 & 1 \end{vmatrix}$$

$$= 3 \cdot 5 \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}$$ (various cleanup row ops, all of "no-change" type)

$$= 15 \cdot 1 = 15.$$

Therem $\det A \neq 0$ iff $\operatorname{rref}(A) = I$ iff A^{-1} exist

already know this equivalence

pf: start with A

do elementary row ops

- change sign if swap rows (original det is opposite of new det.)
- no change if replace row$_i$ by row$_i$ + c row$_k$ $k \neq i$
- if factor "c" out of row, original det is c times det of what's left

$\operatorname{rref}(A)$

if $\operatorname{rref}(A) = I$

$$|A| = \underbrace{c_1 c_2 \cdots c_N}\ \underbrace{|I|}_{1}$$

c_i's are ±1, or other nonzero const's.

and $|A| = c_1 c_2 \cdots c_N \neq 0$!

if $\operatorname{rref}(A) \neq I$

rref(A)$_i$ has a row of 0's.

$$|A| = c_1 c_2 \cdots c_N \begin{vmatrix} \sim\sim\sim \\ 0\,0\,0\cdot 0 \end{vmatrix} = 0$$

∎

Thm $\det(AB) = (\det A)(\det B)$ ← true, but not obvious:

$\det(A+B) \neq \det A + \det B$!!!

$|A|$

rref

$\operatorname{rref}(A) = I$

$$|A| = \underbrace{c_1 \cdots c_N}_{|A|} |I|$$

$\operatorname{rref}(A) \neq I$

$$c_1 \cdots c_N \begin{vmatrix} \sim\sim \\ 0\,0\,0 \end{vmatrix}$$

$|AB|$

do indentical rowops to AB → is same as doing row ops to A, <u>then</u> multiplying

$\operatorname{rref}(A) = I$

$$|AB| = c_1 \cdots c_N |IB| = |A||B|$$

$\operatorname{rref}(A) \neq I$

$$|AB| = c_1 \cdots c_N \left| \begin{pmatrix} \sim\sim \\ 0\,0\,0 \end{pmatrix} B \right| = c_1 \cdots c_N \begin{vmatrix} \sim\sim \\ 0\,0\,0\,0 \end{vmatrix} = 0 = |A||B|.$$

If $B_{m \times n} = [b_{ij}]$ then the <u>transpose</u> of B, B^T is defined by

$$\text{entry}_{ij}(B^T) := \text{entry}_{ji}(B) = b_{ji}$$

the effect is to switch rows & columns (and vise verse)

e.g. $$\text{entry}_{ij}(B^T) := \text{entry}_j(\text{row}_i(B^T))$$
$$\|$$
$$b_{ji} = \text{entry}_j(\text{col}_i(B))$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}^T = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$$

<u>Definition</u> the adjoint matrix of $A = [a_{ij}]$ (write $\text{adj}(A)$) is the transpose of the cofactor matrix $[C_{ij}]$ $\quad \text{adj}(A) = [C_{ij}]^T$

<u>Theorem</u> When A^{-1} exists, $\boxed{A^{-1} = \frac{1}{|A|}\,\text{adj}(A)}$

<u>examples</u>

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad [M_{ij}] = \begin{bmatrix} d & c \\ b & a \end{bmatrix}; \quad [C_{ij}] = \begin{bmatrix} d & -c \\ -b & a \end{bmatrix}; \quad \text{adj}(A) = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

so $A^{-1} = \frac{1}{|A|}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$!

our friend

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 0 & 3 & 1 \\ 2 & -2 & 1 \end{bmatrix}$$

$$[C_{ij}] = \begin{bmatrix} \begin{vmatrix} 3 & 1 \\ -2 & 1 \end{vmatrix} & -\begin{vmatrix} 0 & 1 \\ 2 & 1 \end{vmatrix} & \begin{vmatrix} 0 & 3 \\ 2 & -2 \end{vmatrix} \\ -\begin{vmatrix} 2 & -1 \\ -2 & 1 \end{vmatrix} & \begin{vmatrix} 1 & -1 \\ 2 & 1 \end{vmatrix} & -\begin{vmatrix} 1 & 2 \\ 2 & -2 \end{vmatrix} \\ \begin{vmatrix} 2 & -1 \\ 3 & 1 \end{vmatrix} & -\begin{vmatrix} 1 & -1 \\ 0 & 1 \end{vmatrix} & \begin{vmatrix} 1 & 2 \\ 0 & 3 \end{vmatrix} \end{bmatrix} = \begin{bmatrix} 5 & 2 & -6 \\ 0 & 3 & 6 \\ 5 & -1 & 3 \end{bmatrix}$$

$$\text{adj}(A) = \begin{bmatrix} 5 & 0 & 5 \\ 2 & 3 & -1 \\ -6 & 6 & 3 \end{bmatrix}$$

$$A^{-1} = \frac{1}{15}\begin{bmatrix} 5 & 0 & 5 \\ 2 & 3 & -1 \\ -6 & 6 & 3 \end{bmatrix}$$

check!

proof that $A^{-1} = \underbrace{\frac{1}{|A|}\operatorname{adj}(A)}_{X}$:

does $AX = I$?

$$\text{entry}_{\ell m}(AX) = \sum_{k=1}^{n} a_{\ell k}\, \underbrace{X_{km}}_{\frac{1}{|A|}C_{mk}}$$

$$= \frac{1}{|A|}\sum_{k=1}^{n} a_{\ell k}\, C_{mk}$$

→ if $\ell = m$ this sum is expansion for $|A|$ down $\ell = m^{th}$ col
so $\text{entry}_{mm}(AX) = \frac{|A|}{|A|} = 1$

↘ if $\ell \neq m$ we are expanding a det. down the m^{th} col of a matrix in which we replaced column m with the ℓ^{th} column, so two col's are equal, so det = 0 !

this shows $AX = I$. ■
(which implies $XA = I$).

Cramer's rule

let $\vec{x}$ solve $A\vec{x} = \vec{b}$ for invertible A.

then $$x_k = \frac{\left| c_1 \middle| c_2 \middle| \cdots \middle| b \middle| \cdots c_n \right|}{|A|}$$

numerator is det of matrix obtained from A by replacing ~~row~~ column k by $\vec{b}$.

proof: $x_k = \text{entry}_k(A^{-1}\vec{b})$

$$= \text{entry}_k\left(\frac{1}{|A|}\operatorname{adj}(A)\vec{b}\right)$$

$$= \frac{1}{|A|}\,\text{row}_k(\operatorname{adj}(A)) \cdot \vec{b}$$

$$= \frac{1}{|A|}\underbrace{\sum_{\ell=1}^{n} C_{\ell k}\, b_\ell}$$

this is the expansion of the det in numerator of Cramer's rule, down the k^{th} column!

■

Review sheet for 1st exam, which is Monday 10/4

problem session

Saturday (tomorrow) 10-11:30
EMCB 112

<u>Chapter 1</u> : Methods to solve certain 1st order DE's

- integration, for $\frac{dy}{dx} = f(x)$
  - position, veloc, accel, when accel is a fcn of t alone.
- separable DE's
  - growth & decay (exponential)
    - populations, radioactive decay, Newton's law of cooling (with constant ambient temp.) drug elimination, rumor propogation, disease spread.
    - NO TORRICELLI ON EXAM
- linear 1st order DE's
  - mixing problems
- slope fields and phase diagrams to understand qualitative behavior of sol'ns without knowing formula for solution fcn.
  - How to draw these, especially for autonomous DE's.

<u>Chapter 2</u> : Applications in depth

- population models: logistic, doomsday / extinction, harvesting logistic; understand derivations, how to find solutions (by separating variables), how to plot slope fields & phase portraits, how to find equilibrium solutions and evaluate stability / unstability.
- equilibrium solt'ns & stability for general 1st order autonomous DE's.
- acceleration-velocity models, especially linear drag (force proportional to velocity.)
- NO NUMERICAL METHODS (2.4-2.6) ON EXAM.

<u>Chapter 3</u> : Linear systems and matrices

- solving linear systems by creating the augmented matrix, using elementary row ops to get rref, deducing solution by backsolving.
- geometric meaning of linear systems in 2 or 3 variables
- Matrix algebra : addition, scalar multiplication, matrix multiplication
  - What algebra rules hold, and which one(s) don't.
- Matrix inverses
  - how to compute via row operations
  - how to solve linear systems with the inverse matrix, if the inverse exists.
- Determinants
  - how to compute with cofactors
  - how to compute with row ops
  - Cramer's rule
  - Adjoint formula for the inverse, esp. 2×2 & 3×3 cases.