Math 2250-3
Monday 11/8

We are discussing §5.6 : Forced harmonic oscillators

page 3 Friday : undamped, with letters

After we solve

$$\begin{cases} x'' + \frac{k}{m}x = \frac{F_0}{m}\cos\omega t \\ x(0)=0 \\ x'(0)=0 \end{cases}$$

What is the sol'n to IVP with

$x(0)=x_0$
$x'(0)=v_0$?

page 4 Friday : example 2 is really on page 351, in new text.
see picture in maple notes too

page 5 Friday : A slick way to guess the solution to the resonance IVP

$$\begin{cases} x'' + \frac{k}{m}x = \frac{F_0}{m}\cos\omega_0 t \\ x(0)=0 \\ x'(0)=0 \end{cases}$$

Is to write down the page 3 soln, ($\omega \neq \omega_0$), and let $\omega \to \omega_0$

$$x(t) = \frac{F_0/m}{\omega_0^2-\omega^2}\, \underbrace{2\sin\left(\frac{\omega_0-\omega}{2}t\right)}_{\approx \frac{\omega_0-\omega}{2}t \text{ for } |\omega_0-\omega| \text{ tiny}} \sin\left(\frac{\omega_0+\omega}{2}t\right)$$

$$\longrightarrow \frac{F_0/m}{\omega_0+\omega_0}\, t \sin\left(\frac{\omega_0+\omega_0}{2}t\right) = \boxed{\frac{F_0}{2m\omega_0} t \sin(\omega_0 t)}$$

example on page 5 is really on text page 352

Damped forced oscillations $(c>0)$

$$mx'' + cx' + kx = F_0 \cos\omega t$$

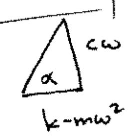
$$
\begin{aligned}
k\,(\quad & x_p = A\cos\omega t + B\sin\omega t \\
+c\,(\quad & x_p' = -A\omega\sin\omega t + B\omega\cos\omega t \\
+m\,(\quad & x_p'' = -A\omega^2\cos\omega t - B\omega^2\sin\omega t
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{L}(x_p) = {} & \cos\omega t\,[kA + Bc\omega - mA\omega^2] \quad = \cos\omega t\,(F_0) \\
& + \sin\omega t\,[kB - Ac\omega - mB\omega^2]
\end{aligned}
$$

$$\begin{bmatrix} k-m\omega^2 & c\omega \\ -c\omega & k-m\omega^2 \end{bmatrix}\begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} F_0 \\ 0 \end{bmatrix}$$

so $$\begin{bmatrix} A \\ B \end{bmatrix} = \frac{1}{(k-m\omega^2)^2 + c^2\omega^2}\begin{bmatrix} k-m\omega^2 & -c\omega \\ c\omega & k-m\omega^2 \end{bmatrix}\begin{bmatrix} F_0 \\ 0 \end{bmatrix}$$

$$\boxed{\begin{bmatrix} A \\ B \end{bmatrix} = \frac{1}{(k-m\omega^2)^2 + c^2\omega^2}\begin{bmatrix} F_0(k-m\omega^2) \\ F_0\, c\omega \end{bmatrix}}$$



So $A\cos\omega t + B\sin\omega t$

$$= C\cos(\omega t - \alpha)$$

$$C = \frac{F_0}{(k-m\omega^2)^2 + c^2\omega^2}\sqrt{(k-m\omega^2)^2 + c^2\omega^2} = \frac{F_0}{\sqrt{\underbrace{(k-m\omega^2)^2}_{\text{small if } \omega \approx \omega_0} + \underbrace{c^2\omega^2}_{\text{small if } c \approx 0}}}$$

$$x_H(t) = \begin{cases} c_1 e^{r_1 t} + c_2 e^{r_2 t} & r_1 < r_2 < 0 \quad \text{undendamped.} \\ e^{-pt}(c_1 + c_2 t) & \text{critically damped} \\ e^{-pt}(A\cos\omega_1 t + B\sin\omega_1 t) & \text{undendamped.} \end{cases}$$

So, full soltn

$$x(t) = \underbrace{x_p(t)}_{\text{periodic}} + \underbrace{x_H(t)}_{\substack{\text{decays exponentially do zero} \\ \text{regardless of choices of } C_1, C_2 \text{ (or } A, B).}}$$

So we call $x_p(t) = C\cos(\omega t - \alpha) = x_{sp}(t)$; steady periodic

$x_H(t) = x_{tr}(t)$; transient sol'n, reflects initial values

<u>example</u>

$$\begin{cases} x'' + 2x' + 26x = 82\cos 4t \\ x(0) = 6 \\ x'(0) = 0 \end{cases}$$

could redo page 2 with numbers rather than letters (exam possibility), but we can also plug in:

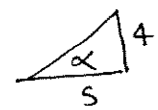
$$x_p(t) = A\cos 4t + B\sin 4t$$

$m = 1$
$\omega = 4$
$c = 2$
$k = 26$

$$\begin{bmatrix} A \\ B \end{bmatrix} = \frac{1}{100+64}\begin{bmatrix} 82\cdot 10 \\ 82\cdot 8 \end{bmatrix} = \begin{bmatrix} \frac{820}{164} \\ \frac{656}{164} \end{bmatrix} = \begin{bmatrix} 5 \\ 4 \end{bmatrix}$$

$k - m\omega^2 = 26 - 16 = 10$
$c\omega = 8$
$F_0 = 82$

$$x_p(t) = 5\cos 4t + 4\sin 4t = \sqrt{41}\cos(4t - \alpha)$$



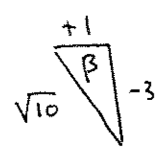
$\alpha = \arctan(.8)$.

$x_H(t)$:

$$r^2 + 2r + 26 = 0$$
$$(r+1)^2 + 25 = 0$$
$$(r+1+5i)(r+1-5i) = 0$$
$$r = -1 \pm 5i$$
$$x_H(t) = e^{-t}(A\cos 5t + B\sin 5t)$$

$$x(t) = 5\cos 4t + 4\sin 4t + e^{-t}(A\cos 5t + B\sin 5t).$$

$$\left.\begin{aligned} x(0) &= 6 = 5 + A \Rightarrow A = 1 \\ x'(0) &= 0 = 0 + 16 - A + 5B = 15 + 5B \Rightarrow B = -3 \end{aligned}\right\}$$



$$x(t) = \underbrace{\sqrt{41}\cos(4t - \alpha)}_{x_{sp}(t)} + \underbrace{e^{-t}\sqrt{10}\cos(5t - \beta)}_{x_{tr}(t).}$$

$\alpha = \arctan(.8)$ $\qquad$ $\beta = \arctan(-3)$

Practical resonance (see page 2)

$$x_{sp}(t) = \frac{F_0}{\sqrt{(k-m\omega^2)^2 + c^2\omega^2}} \cos(\omega t - \alpha)$$

$$\left.\begin{array}{l} \text{if } c \text{ is small} \\ \text{and } \omega^2 \approx \omega_0^2 = \frac{k}{m} \end{array}\right\} \text{denom tiny} \Rightarrow \text{amplitude of } x_{sp} \text{ huge (relative to } F_0\text{)}$$

This is called <u>practical resonance</u>

Vary ω in page 3 example

$$x'' + cx' + 26x = 82\cos\omega t$$

$$C = \frac{82}{\sqrt{(26-\omega^2)^2 + c^2\omega^2}} = C(\omega)$$

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> plot(82/sqrt((26-omega^2)^2+4*omega^2),omega=0..20,
  color=black, title='practical resonance, figure5.6.9 page357');
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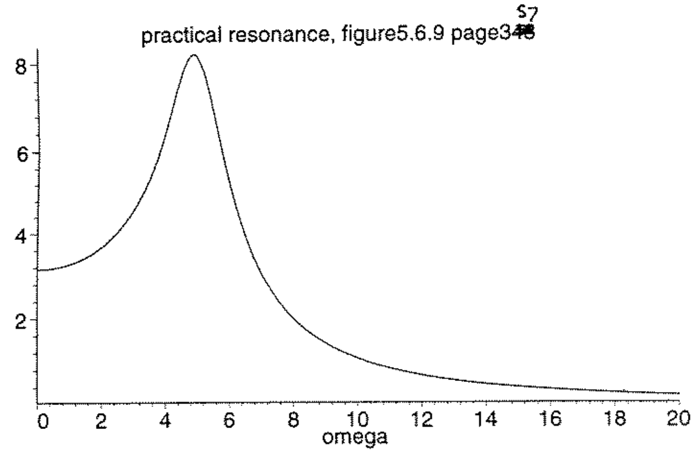


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> plot(82/sqrt((26-omega^2)^2+.01*omega^2),omega=0..20,
  color=black, title='practical resonance with c=0.1');
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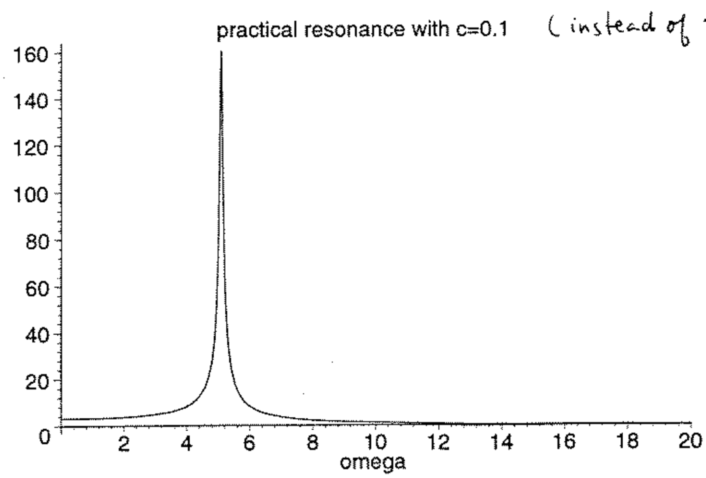

(instead of 4)

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