Math 2250-3
Friday 11/5

I will be in PClab 1735 (basement Marriott) Saturday 10:10 - 11:40 a.m. to answer Maple questions

Forced harmonic oscillators Overview

$$mx'' + cx' + kx = F_0 \cos\omega t$$

- undamped $(c=0)$

$$mx'' + kx = F_0 \cos\omega t$$

- $\omega \neq \omega_0 = \sqrt{\frac{k}{m}}$ $\quad x_p = A\cos\omega t + B\sin\omega t = C\cos(\omega t - \alpha)$

general soltn
$x = x_p + x_H = C\cos(\omega t - \alpha) + C_0\cos(\omega_0 t - \alpha_0)$

- $\omega = \omega_0$ $\quad x_p = t(A\cos\omega_0 t + B\sin\omega_0 t) = t\,C\cos(\omega_0 t - \alpha)$ $\quad$ general soltn $x = x_p + x_H$

RESONANCE!

- $\omega \neq \omega_0$, but ω close to ω_0
BEATING

- damped $(c>0)$ (More details Monday 11/8)

$$x_p = A\cos\omega t + B\sin\omega t = C\cos(\omega t - \alpha)$$

$x_H \rightarrow$ 3 possibilities (over, critically, underdamped)
key feature they all share is $x_H(t) \rightarrow 0$ exponentially fast as $t \rightarrow \infty$.

so x_p is called x_{sp} (steady periodic)
x_H is called x_{tr} (transient)

when damping c is small, and ω is near ω_0, then amplitude C of x_{sp} will be a large multiple of forcing amplitude F_0. This is called
approximate resonance

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$$\begin{cases} x'' + 9x = 80\cos 5t \\ x(0) = 0 \\ x'(0) = 0 \end{cases}$$

x_H:

x_p:

soltn we should get is

$x(t) = 5\cos 3t - 5\cos 5t$

$= 5(\cos 3t - \cos 5t)$

$= 10\sin(4t)\sin(t)$

period = ?

$\cos[(a-b)t] - \cos[(a+b)t]$

$= \cos at \cos bt + \sin at \sin bt$
$\quad - \cos at \cos bt + \sin at \sin bt$

$= 2\sin(at)\sin(bt)$

$a=4$
$b=1$

undamped, cont'd, with letters:
$\omega \neq \omega_0$

$$\begin{cases} x'' + \frac{k}{m} x = \frac{F_0}{m} \cos \omega t \\ x(0) = 0 \\ x'(0) = 0 \end{cases}$$

$$\begin{aligned} k/m \,(\quad & x_p = A \cos \omega t \\ & x_p' = -A\omega \sin \omega t \\ +1 \,(\quad & x_p'' = -A\omega^2 \cos \omega t \end{aligned}$$

$$L(x_p) = \cos \omega t \, [A] \Big[-\omega^2 + \overbrace{\tfrac{k}{m}}^{\omega_0^2}\Big]$$

$$= \cos \omega t \left[\frac{F_0}{m}\right]$$

deduce $A(\omega_0^2 - \omega^2) = \frac{F_0}{m}$

$$\boxed{A = \frac{F_0/m}{\omega_0^2 - \omega^2}}$$

$$x_p(t) = \frac{F_0/m}{\omega_0^2 - \omega^2} \cos \omega t$$

$$x(t) = x_p(t) + x_H(t)$$

$$= \frac{F_0/m}{\omega_0^2 - \omega^2} \cos \omega t + A \cos \omega_0 t + B \sin \omega_0 t$$

for our IVP observe $B = 0$, $A = \dfrac{-F_0/m}{\omega_0^2 - \omega^2}$

$$x(t) = \frac{F_0/m}{\omega_0^2 - \omega^2} \left[\cos \omega t - \cos \omega_0 t\right]$$

$$\boxed{x(t) = \frac{F_0/m}{\omega_0^2 - \omega^2} \, 2 \sin\left(\left(\frac{\omega_0 - \omega}{2}\right)t\right) \sin\left(\left(\frac{\omega_0 + \omega}{2}\right)t\right)}$$

$\overline{\omega} = \frac{\omega_0 + \omega}{2}$, $\underline{\omega} = \frac{\omega_0 - \omega}{2}$

so $\omega = \overline{\omega} - \underline{\omega}$
$\omega_0 = \overline{\omega} + \underline{\omega}$

example 2 p 342 Beating

IVP soltn pg 3:

$$x(t) = \frac{F_0/m}{\omega_0^2-\omega^2}\, 2\sin\left(\frac{\omega_0-\omega}{2}t\right)\sin\left(\frac{\omega_0+\omega}{2}t\right)$$

$\omega \approx \omega_0$ this is slowly varying; period $\frac{4\pi}{|\omega_0-\omega|}$

$\omega \approx \omega_0$ then $\bar{\omega} \approx \omega_0$ too

e.g. $m = .1$
$F_0 = 50$
$\omega_0 = 55$
$\omega = 45$

$$x(t) = \frac{500}{(55-45)(55+45)}\, 2\sin 5t \sin 50t$$

$$= \sin 5t \sin 50t$$

resonance $\omega = \omega_0$

$$x'' + \omega_0^2 x = \frac{F_0}{m} \cos\omega_0 t$$

$\omega_0^2($ try $x_p = t(A\cos\omega_0 t + B\sin\omega_0 t)$

$x_p' = t(-A\omega_0 \sin\omega_0 t + B\omega_0\cos\omega_0 t) + A\cos\omega_0 t + B\sin\omega_0 t$

$1($ $x_p'' = t(-A\omega_0^2\cos\omega_0 t - B\omega_0^2 \sin\omega_0 t) + [-A\omega_0\sin\omega_0 t + B\omega_0\cos\omega_0 t]2$

$$L(x_p) = t(0) + 2[-A\omega_0\sin\omega_0 t + B\omega_0\cos\omega_0 t]$$

want $= \frac{F_0}{m}\cos\omega_0 t$;

so $A = 0$

$2B\omega_0 = F_0/m$

$B = \frac{F_0}{2\omega_0 m}$

$$\boxed{x_p(t) = \frac{F_0}{2m\omega_0} t \sin\omega_0 t}$$

note, this solves IVP with

$x(0) = 0$

$x'(0) = 0$

or: variation of params

$y_1 = \cos\omega_0 t$

$y_2 = \sin\omega_0 t$ $\quad y = u_1 y_1 + u_2 y_2$

(try it!)

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$m = 1$

$\omega_0 = 50$

$F_0 = 100$

$x(t) = \frac{100}{100} t\sin 50t$

$= t\sin 50t$