Math 2250-3
Wed 11/3

HW for Wed 11/10
(in addition to MAPLE)

5.5 3 4 12 13, 19, 34, 37, 43, 49 50, 52
5.6 3, 5, 7, 14, 16, 21, 22

§5.5 : Finding y_p's for $L(y)=f$.

Recall, if L is a linear operator the general sol'n to

$$L(y)=f$$

is $$y = y_p + y_H$$

where y_p is any particular sol'n and y_H is the general (n-dim'l) solution to $L(y)=0$

How to find y_p's.

Book calls the method "undetermined coefficients", for constant coefficient linear DE's, is really just "guessing" (but could be justified with vector space theory).

<u>example 1</u>

$$L(y) = y'' + 3y' + 4y$$

solve $L(y) = 3x+2$

<u>try</u> $y_p = Ax + B$ (because when you apply L to polys of degree ≤ 1 you get back polys of degree ≤ 1).

$$\begin{aligned} 4\,(\quad & y_p = Ax + B \\ +3\,(\quad & y_p' = A \\ +1\,(\quad & y_p'' = 0 \end{aligned}$$

$$L(y_p) = x(4A) + 1(3A+4B) = 3x+2$$

$$\left.\begin{aligned} 4A &= 3 \\ 3A + 4B &= 2 \end{aligned}\right\} \quad \begin{aligned} A &= 3/4 \\ \tfrac{9}{4} + 4B &= 2 \\ 4B &= -\tfrac{1}{4} \\ B &= -\tfrac{1}{16} \end{aligned}$$

$$\boxed{y_p = \frac{3}{4}x - \frac{1}{16}}$$

y_H: $r^2 + 3r + 4 = 0$

$$r = \frac{-3 \pm \sqrt{9-16}}{2} = -\frac{3}{2} \pm i\frac{\sqrt{7}}{2}$$

$$\boxed{y_H(x) = e^{-3/2\,x}\left(A\cos\frac{\sqrt{7}}{2}x + B\sin\frac{\sqrt{7}}{2}x\right)}$$

$$\boxed{y(x) = \frac{3}{4}x - \frac{1}{16} + e^{-3/2\,x}\left(A\cos\frac{\sqrt{7}}{2}x + B\sin\frac{\sqrt{7}}{2}x\right)}$$

example 2 $y'' - 4y = 2e^{3x}$

find y_p:

try $y_p = Ae^{3x}$

$$L(e^{3x}) = 9e^{3x} - 4e^{3x} = 5e^{3x}$$

so $L(Ae^{3x}) = 5Ae^{3x}$
$$\qquad\qquad \| $$
$$\qquad\qquad 2; \qquad A = \tfrac{2}{5} \qquad y_p = \tfrac{2}{5}e^{3x}$$

example 3 $y'' - 4y = 10e^{2x}$

find y_p

try

oh oh. → the RHS was related to y_H. There's a way to fix your guess:

example 4 find y_p to $y'' - 4y = 4e^{3x} + 5e^{2x}$

ans: (hint: look at the 2 examples above) guessing rules

If $L(y)$ is constant coeff. linear operator, what would be your first guess at the form of y_p if you wanted to solve

$L(y) = 3\cos 2x$

$y_p =$

$L(y) = 4e^{2x}\sin 3x$

$y_p =$

$L(y) = x^3 + 6x^2 - 5$

$y_p =$

$L(y) = 7e^{13x}$

$y_p =$

$L(y) = x\sin 2x$

$y_p =$

there are special rules if $f(x) =$ RHS is related to y_H: you multiply your guess for y_p by x^s, where s is the lowest natural power making each term in your guess NOT a solution of the homog. eqtn

See table page 341

If guessing won't work (maybe your linear DE is not const coeff) there is a method, variation of parameters, that will.

illustrate method on 2^{nd} example page 2:

$$y'' - 4y = 10e^{2x}$$

[by improved "guessing" we got

$$y_p = \frac{5}{2} x e^{2x}$$
$$y = \frac{5}{2} x e^{2x} + c_1 e^{2x} + c_2 e^{-2x}].$$

Variation of parameters: to find y_p if you have a basis $\{y_1, y_2\}$ for y_H

$$\mathcal{L}(y) = y'' + p(x) y' + q(x) y$$

$y_H = \text{span}\{y_1, y_2\}$ known.

to solve

$$\mathcal{L}(y) = f$$

try $y_p = u_1 y_1 + u_2 y_2$ $\quad u_1, u_2 = $ fcns of x

$$\begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix} \begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = \begin{bmatrix} 0 \\ f \end{bmatrix}$$

$q(x)\ (\quad y_p = u_1 y_1 + u_2 y_2$

$p(x)\ (\quad y_p' = u_1 y_1' + u_2 y_2' + \underbrace{u_1' y_1 + u_2' y_2}_{\text{set } \equiv 0}$

$1\ (\Rightarrow y_p'' = u_1 y_1'' + u_2 y_2'' + \underbrace{u_1' y_1' + u_2' y_2'}_{\text{set } = f}$

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so $\mathcal{L}(y_p) = u_1 \underbrace{\mathcal{L}(y_1)}_{0} + u_2 \underbrace{\mathcal{L}(y_2)}_{0} + f$

$= f$

$$\begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = \frac{1}{W} \begin{bmatrix} y_2' & -y_2 \\ -y_1' & y_1 \end{bmatrix} \begin{bmatrix} 0 \\ f \end{bmatrix}$$

$$\begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = \frac{1}{W} \begin{bmatrix} -y_2 f \\ y_1 f \end{bmatrix}$$

integrate to get u_1 & u_2 then get y_p.

$y_1 = e^{2x}$

$y_2 = e^{-2x}$

$$W = \begin{vmatrix} e^{2x} & e^{-2x} \\ 2e^{2x} & -2e^{-2x} \end{vmatrix} = -4$$

$f = 10e^{2x}$

$$u_1' = -\frac{1}{4}\left(-e^{-2x} 10 e^{2x}\right) = \frac{5}{2}$$
$$u_2' = -\frac{1}{4}\left(e^{2x} 10 e^{2x}\right) = -\frac{5}{2} e^{4x}$$

take $u_1 = \frac{5}{2} x$

$u_2 = -\frac{5}{8} e^{4x}$

$$y_p = u_1 y_1 + u_2 y_2$$
$$= \frac{5}{2} x e^{2x} - \frac{5}{8} e^{4x} e^{-2x}$$

$\boxed{y_p = \frac{5}{2} x e^{2x} - \frac{5}{8} e^{2x}}$, (agrees since $-\frac{5}{8} e^{2x}$ is a y_H)