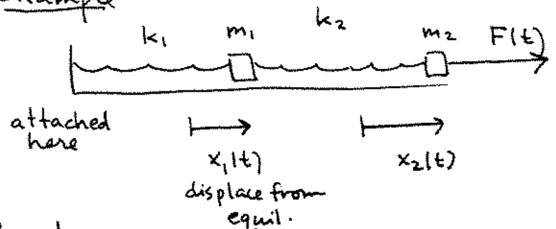
Math 2250-3
Wed 11/17 §7.1-7.2

We saw Monday how a system of tanks with interconnecting pipes can lead to a 1^{st} order system of linear DE's
(and we used eigenvalues & eigenvects to solve the system of DE's!)

Example



Newton:

$$m_1 x_1''(t) = \underbrace{k_2(x_2 - x_1)}_{\substack{2^{nd}\text{ spring is stretched}\\ \text{this much}}} - k_1 x_1$$

$$m_2 x_2''(t) = -k_2(x_2 - x_1) + F(t)$$

e.g. $\left.\begin{array}{l} m_1 = 2 \\ m_2 = 1 \\ k_1 = 4 \\ k_2 = 2 \\ f(t) = 40\sin 3t \end{array}\right\} \Rightarrow \left\{\begin{array}{l} 2x_1'' = 2(x_2 - x_1) - 4x_1 = -6x_1 + 2x_2 \\ x_2'' = -2(x_2 - x_1) + 40\sin 3t = 2x_1 - 2x_2 + 40\sin 3t \end{array}\right.$

or $\begin{bmatrix} x_1'' \\ x_2'' \end{bmatrix} = \begin{bmatrix} -3 & 1 \\ 2 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 40\sin 3t \end{bmatrix}$

- What's a natural IVP for this 2^{nd} order system of 2 linear DE's?
(We won't solve it today – see §7.4)

note: Any system of DE's can be converted into an equivalent system of 1^{st} order DE's, by introducing extra unknown functions.

(1) $\boxed{\begin{array}{l} x_1'' = -3x_1 + x_2 \\ x_2'' = 2x_1 - 2x_2 + 40\sin 3t \end{array}}$

let $y_1 = x_1'$
$y_2 = x_2'$

(2) $\boxed{\begin{array}{l} x_1' = y_1 \\ y_1' = -3x_1 + x_2 \\ x_2' = y_2 \\ y_2' = 2x_1 - 2x_2 + 40\sin 3t \end{array}}$

solutions of (1) yield solutions of (2) and vise verse.

This correspondence is sometimes useful.

<u>Example</u> : Study the correspondence between n^{th} order eqtns or systems and 1^{st} order ones :

$x = x(t)$

<u>2^{nd} order linear homog DE</u>

$$x'' - x' - 2x = 0$$

$x = e^{rt}$

$p(r) = r^2 - r - 2 = 0$

$(r-2)(r+1) = 0$

$$\boxed{x_H(t) = c_1 e^{2t} + c_2 e^{-t}}$$

so we could immediately solve the 1^{st} order system :

$x_H(t) = c_1 e^{2t} + c_2 e^{-t}$

$x_H'(t) = 2c_1 e^{2t} - c_2 e^{-t}$

so $\begin{bmatrix} x \\ y \end{bmatrix} = c_1 e^{2t} \begin{bmatrix} 1 \\ 2 \end{bmatrix} + c_2 e^{-t} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

Also, see work at right!

<u>IVP</u> $\begin{cases} x'' - x' - 2x = 0 \\ x(0) = 2 \\ x'(0) = 1 \end{cases}$

$c_1 + c_2 = 1$

$2c_1 - c_2 = 1$

$$\boxed{x(t) = e^{2t} + e^{-t}}$$

<u>1^{st} order system of 2 DE's</u>

$y = x'$

$x' = y$

$y' = y + 2x$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}.$$

try $\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = e^{\lambda t} \vec{v}$

for $\vec{x}'(t) = A\vec{x}$

$\lambda e^{\lambda t} \vec{v} = A e^{\lambda t} \vec{v}$

$\lambda \vec{v} = A\vec{v}$ $\quad$ $\vec{v}$ eigenvect of A

$0 = (A - \lambda I)\vec{v}$ $\quad$ eigenval λ

$$\det \begin{bmatrix} -\lambda & 1 \\ 2 & 1-\lambda \end{bmatrix} = \lambda^2 - \lambda - 2 = p(\lambda)$$

does this "characteristic polynomial" look familiar?

$= (\lambda - 2)(\lambda + 1)$

$\lambda = 2$

$\begin{array}{cc|c} -2 & 1 & 0 \\ 2 & -1 & 0 \end{array}$

$\vec{v} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$\lambda = -1$

$\begin{array}{cc|c} 1 & 1 & 0 \\ 2 & 2 & 0 \end{array}$

$\vec{v} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

eigenbasis

$$\Rightarrow \begin{pmatrix} x \\ y \end{pmatrix} = c_1 e^{2t} \begin{bmatrix} 1 \\ 2 \end{bmatrix} + c_2 e^{-t} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

✓

IVP $\begin{cases} \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \\ \begin{bmatrix} x(0) \\ y(0) \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \end{cases}$

$\begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

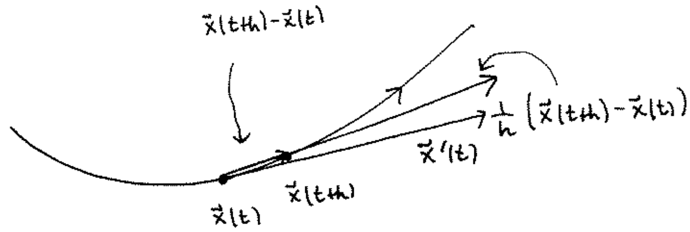
$\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = e^{2t} \begin{bmatrix} 1 \\ 2 \end{bmatrix} + e^{-t} \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

Geometric interpretation of 1^{st} order system of DE's:

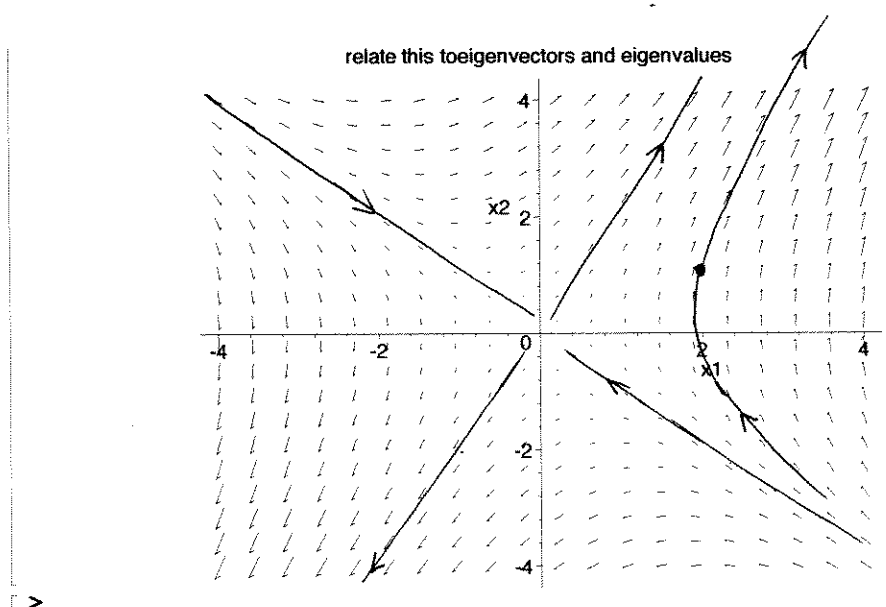
IVP $\begin{cases} \dfrac{d\vec{x}}{dt} = \vec{F}(t,\vec{x}) \\ \vec{x}(t_0) = \vec{x}_0 \end{cases}$

- Recall, if $\vec{x}(t)$ is parametric curve in space (think partical position at time t) then $\vec{x}'(t)$ is the tangent (or velocity) vector:



So the IVP says you know where you start $(\vec{x}(t_0)=\vec{x}_0)$, and you know your velocity vector (depending on time & location) $\longrightarrow$ so you expect a unique sol'n

page 2 example continued :



$$\begin{cases} \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \\ \begin{bmatrix} x(0) \\ y(0) \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \end{cases}$$

The Harmonic oscillator
(unforced, undamped).

$$x'' + x = 0$$

$$x' = y$$

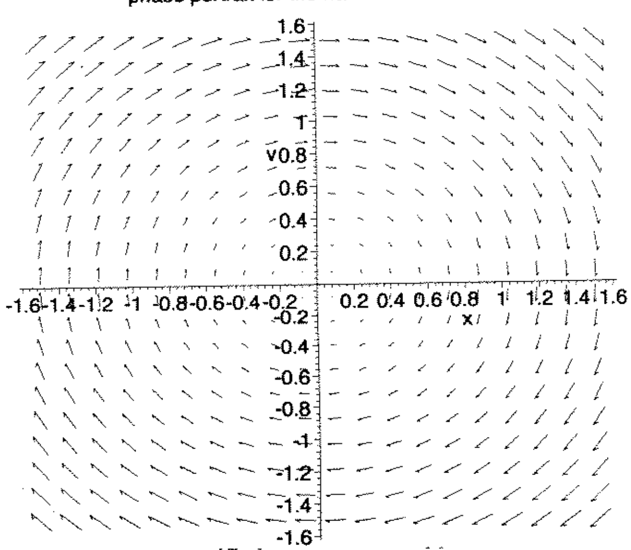
$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} y \\ -x \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$x(t) = C\cos(t-\alpha)$$

so $\begin{bmatrix} x \\ x' \end{bmatrix} = C \begin{bmatrix} \cos(t-\alpha) \\ -\sin(t-\alpha) \end{bmatrix}$.

clockwise circles!

phase portrait for the harmonic oscillator



$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} y \\ -x \end{bmatrix}$$

what if you try solving this 1st order system of DE's with the eigenvalue - eigenvector method?