Math 2250-3
Monday 11/15

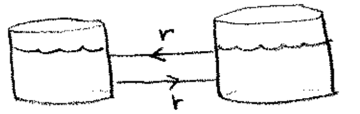
HW for Wed 11/24

6.1 7, 17, 18, 24, 29
7.1 1, 3 5 8 11, 18, 21, 24, 26
7.2 5, 9, 10, 13, 14, 16, 21, 23, 25
7.3 3, 4, 6 13, 14, 25, 30, 32, 37

§6.1 A return to linear algebra, so we can tackle the next step in DE's:

1st order DE's
↓
n^{th} order linear DE's
in particular, springs
↓
systems of 1st & 2nd order DE's
e.g. tank systems & spring systems

example:



$V_1 = 100$ g $\quad V_2 = 50$ g

$x_1(t) \quad x_2(t)$

$r = 10$ g/min

$$\frac{dx_1}{dt} = -10\frac{x_1}{100} + 10\frac{x_2}{50}$$

$$\frac{dx_2}{dt} = 10\frac{x_1}{100} - 10\frac{x_2}{50}$$

in matrix form, and IVP:

$$\begin{cases} \begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} -.1 & .2 \\ .1 & -.2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \\ \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \end{cases}$$

This is an example of a 1st order, constant coefficient, homogeneous linear system of DE's.

$$\begin{cases} \vec{x}'(t) = A\vec{x}(t) \\ \vec{x}(t_0) = \vec{x}_0 \end{cases}$$

Solution space will be n-dim'l, and we will look for basis fcns of the form

$x(t) = e^{\lambda t}\vec{v}$ (where $\vec{v}$ is a constant vector)

plug it in to $\vec{x}' = A\vec{x}$:

$$\lambda e^{\lambda t}\vec{v} = Ae^{\lambda t}\vec{v} = e^{\lambda t}A\vec{v}$$

divide by $e^{\lambda t}$: $\boxed{\lambda\vec{v} = A\vec{v}}$ ← $\vec{v}$ is an <u>eigenvector</u> of A with <u>eigenvalue</u> λ

In tank example $A = \begin{bmatrix} -.1 & .2 \\ .1 & -.2 \end{bmatrix}$

Note $\begin{bmatrix} -.1 & .2 \\ .1 & -.2 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} = 0\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ $\qquad \vec{v} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, $\lambda = 0$, soltn $1\begin{bmatrix} 2 \\ 1 \end{bmatrix}$

$\begin{bmatrix} -.1 & .2 \\ .1 & -.2 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} -.3 \\ .3 \end{bmatrix} = -.3\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ $\qquad \vec{v} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$, $\lambda = -.3$, soltn $e^{-.3t}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$

<u>Notice</u> the solutions to the homogeneous system

$$\frac{d\vec{x}}{dt} = A\vec{x}$$

form a <u>subspace</u>

check:

what happens as $t \to \infty$?

so $\vec{x}(t) = c_1\begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2 e^{-.3t}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ solves tank problem.

to solve IVP: $\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$, solve the system $\begin{bmatrix} 2 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$ $\blacksquare$

How to find eigenvalues and eigenvectors?

$A\vec{v} = \lambda\vec{v}$ $\qquad$ ($\vec{v} \neq \vec{0}$ is interesting case)

$A\vec{v} - \lambda\vec{v} = \vec{0}$

$(A - \lambda I)\vec{v} = \vec{0}$

in order for there to be nontrivial solutions to this matrix eqtn, the matrix $A - \lambda I$ must be singular (non-invertible)

iff $\det(A - \lambda I) = 0$

polynomial in λ, called <u>characteristic polynomial</u>

<u>Algorithm</u> for finding evals/evects:

1. Find roots of characteristic polynomial
2. For each root λ find an eigenspace basis, i.e. basis for soltn space to $(A - \lambda I)\vec{v} = \vec{0}$.

<u>Example</u> $A = \begin{bmatrix} -.1 & .2 \\ .1 & -.2 \end{bmatrix}$

$$|A-\lambda I| = \begin{vmatrix} -.1-\lambda & .2 \\ .1 & -.2-\lambda \end{vmatrix} = (\lambda+.1)(\lambda+.2) - .02$$

$$= \lambda^2 + .3\lambda + \cancel{.02} - \cancel{.02}$$

$$= \lambda(\lambda+.3)$$

$$\lambda = 0, -.3$$

$\lambda = 0$ espace:

$$\begin{array}{cc|c} -.1 & .2 & 0 \\ .1 & -.2 & 0 \\ \hline -1 & 2 & 0 \\ 0 & 0 & 0 \end{array}$$

$v_2 = t$
$v_1 = 2t$
$\vec{v} = t\begin{bmatrix} 2 \\ 1 \end{bmatrix}$

basis $\left\{ \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right\}$

$\lambda = -.3$ espace

$$\begin{array}{cc|c} .2 & .2 & 0 \\ .1 & .1 & 0 \\ \hline 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}$$

$v_2 = t$
$v_1 = -t$
$\vec{v} = t\begin{bmatrix} -1 \\ 1 \end{bmatrix}$

basis $\left\{ \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}$

<u>Example</u> $A = \begin{bmatrix} 4 & -2 & 1 \\ 2 & 0 & 1 \\ 2 & -2 & 3 \end{bmatrix}$

$\lambda(-\lambda^2+7\lambda-12)$
$\lambda(\lambda-3)(4-\lambda)$

$$|A-\lambda I| = \begin{vmatrix} 4-\lambda & -2 & 1 \\ 2 & -\lambda & 1 \\ 2 & -2 & 3-\lambda \end{vmatrix} = (4-\lambda)(-\lambda)(3-\lambda) - 4 - 4 + 2\lambda + 2(4-\lambda) + 4(3-\lambda)$$

$$= \lambda^3(-1) + \lambda^2(7) + \lambda(-12+2-2-4) + 1(12)$$

$$= -\lambda^3 + 7\lambda^2 - 16\lambda + 12$$

possible integer roots: $\pm1, \pm2, \pm3, \pm4, \pm6$

$p(2) = -8 + 28 - 32 + 12 = 0$!

so $\lambda - 2$ is a factor

$$\begin{array}{r} -\lambda^2 + 5\lambda - 6 \\ \lambda-2 \,\big)\, \overline{-\lambda^3 + 7\lambda^2 - 16\lambda + 12} \\ -\lambda^3 + 2\lambda^2 \qquad\qquad \\ \hline 5\lambda^2 - 16\lambda \qquad \\ 5\lambda^2 - 10\lambda \qquad \\ \hline -6\lambda + 12 \\ -6\lambda + 12 \\ \hline 0 \end{array}$$

$$p(\lambda) = (\lambda-2)(-1)(\lambda^2 - 5\lambda + 6)$$

$$= (\lambda-2)(\lambda-2)(\lambda-3)$$

$$= (\lambda-2)^2(\lambda-3)$$

$\lambda = 2$ espace:

$$\left[\begin{array}{ccc|c} 2 & -2 & 1 & 0 \\ 2 & -2 & 1 & 0 \\ 2 & -2 & 1 & 0 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 2 & -2 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & -1 & 1/2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$v_3 = t$
$v_2 = s$
$v_1 = s - \frac{1}{2}t$

$$\vec{v} = s\begin{bmatrix}1\\1\\0\end{bmatrix} + t\begin{bmatrix}-\frac{1}{2}\\0\\1\end{bmatrix}$$

basis $\left\{ \begin{bmatrix}1\\1\\0\end{bmatrix}, \begin{bmatrix}-1\\0\\2\end{bmatrix} \right\}$

$\lambda = 3$ espace

$$\left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 2 & -3 & 1 & 0 \\ 2 & -2 & 0 & 0 \end{array}\right]$$

$$\begin{array}{c} \\ -2R_1+R_2 \\ -2R_1+R_3 \end{array} \left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 2 & -2 & 0 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\begin{array}{c} 2R_2+R_1 \\ \\ \\ \end{array} \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$v_3 = t$
$v_2 = t$
$v_1 = t$

$$\vec{v} = t\begin{bmatrix}1\\1\\1\end{bmatrix}$$

basis $\left\{ \begin{bmatrix}1\\1\\1\end{bmatrix} \right\}$.

it will be good, for solving our systems of 1st order homog DE's, if we can amalgamate our eigenspace bases to get a basis for $\mathbb{R}^n$. That happens in this example, since

$$\left\{ \begin{bmatrix}1\\1\\0\end{bmatrix}, \begin{bmatrix}-1\\0\\2\end{bmatrix}, \begin{bmatrix}1\\1\\1\end{bmatrix} \right\} \text{ is an } \mathbb{R}^3 \text{ basis.}$$

Why?

[Such matrices A will be called "diagonalizable"].