Math 2250-3

Monday 11/1

§5.4 cont'd.

Postpone §5.5 HW until next week → only hand in §5.3 & 5.4 this Wed 11/3.

We are studying unforced mechanical vibrations

$$m x'' + c x' + k x = 0$$

On Friday we studied

<u>Case 1</u> Free undamped motion ($c = 0$)

$$m x'' + k x = 0$$

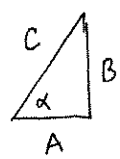
$$x'' + \omega_0^2 x = 0 \qquad \omega_0 := \sqrt{k/m}$$

sol'n

$$x(t) = A \cos \omega_0 t + B \sin \omega_0 t$$

$$= C \cos(\omega_0 t - \alpha)$$

↑ amplitude ↑ phase



C = amplitude

ω_0 = angular freq (rad/time)

$f = \omega_0/2\pi$ (cycles/sec. (hertz)) frequency

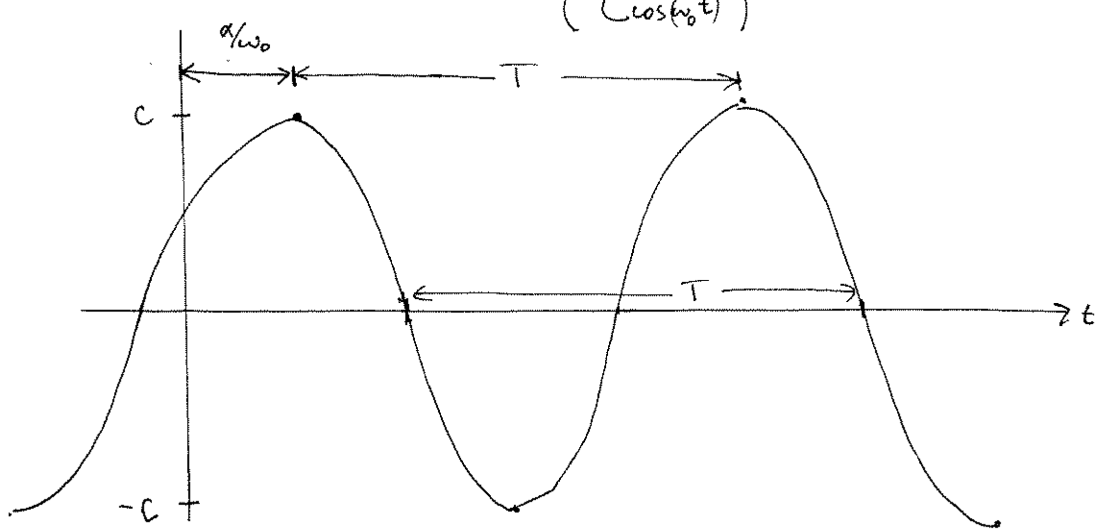
$T = 2\pi/\omega_0$ (time/cycles) period

α = phase angle;

since $\omega_0 t - \alpha = \omega_0(t - \alpha/\omega_0)$

this shifts standard cos curve to the right by α/ω_0

($C\cos(\omega_0 t)$)



Example 1 (page 325)

$m = \frac{1}{2}$ kg

when $x = 2m$, force of spring is 100 N.

i.e. $2k = 100 \quad k = 50$ N/m

$$\frac{1}{2}x'' + 50x = 0$$

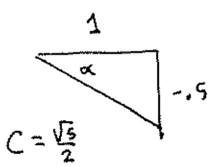
$$x'' + 100x = 0 \qquad \omega_0 = 10$$

IVP $\begin{cases} x_0 = 1 & m \\ v_0 = -5 & m/s. \end{cases}$

$$x(t) = A\cos 10t + B\sin 10t$$

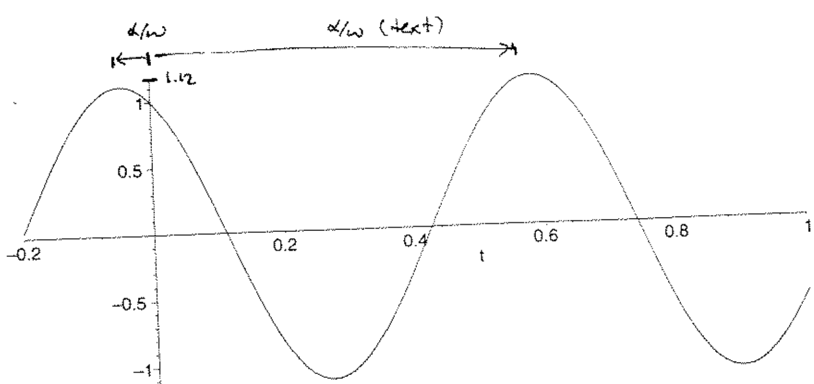
$$\boxed{x(t) = \cos 10t - .5\sin 10t}$$

$$x(t) = \frac{\sqrt{5}}{2}\cos(10t - \alpha) \approx (1.12)\cos(10t + .46)$$



$\alpha = \tan^{-1}(-.5)$
$\approx -.46$ rad.

(compare with text page 326,
says $\alpha = 5.8195$.
both answers are correct!)



Case 2 : Damping

$$m x'' + c x' + k x = 0$$

$$x'' + 2p x' + \omega_0^2 x = 0 \qquad \omega_0 = \sqrt{\tfrac{k}{m}} \text{ still; } \quad \tfrac{c}{m} = 2p; \quad p = \tfrac{c}{2m}$$

$$p(r) = r^2 + 2pr + \omega_0^2 = 0$$

$$r = \frac{-2p \pm \sqrt{4p^2 - 4\omega_0^2}}{2}$$

$$\boxed{r = -p \pm \sqrt{p^2 - \omega_0^2}}$$

Overdamped : $p^2 - \omega_0^2 > 0 \quad (c^2 > 4km)$

$$r = r_1 < r_2 < 0$$

$$x(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t} = e^{r_1 t}\left(c_1 + c_2 e^{(r_2 - r_1)t}\right)$$

Figure 5.4.7 p 327

$\rightarrow$ sol's decay exponentially to zero, cross t-axis at most 1 time.

Critically damped $p^2 - \omega_0^2 = 0 \quad (c^2 = 4km)$

$r = -p$ double root

$$x(t) = c_1 e^{rt} + c_2 t e^{rt} = e^{rt}(c_1 + c_2 t)$$

Figure 5.4.8 p 327 $\rightarrow$ sol's decay exponentially to zero, cross t-axis at most once

Underdamped $p^2 - \omega_0^2 < 0$

set $\omega_1 = \sqrt{\omega_0^2 - p^2} < \omega_0$

$$r = -p \pm i\omega_1$$

$$e^{rt} = e^{(-p \pm i\omega)t}$$

$$x(t) = e^{-pt}(A\cos\omega_1 t + B\sin\omega_1 t)$$

$$= e^{-pt}\left(C\cos(\omega_1 t - \alpha)\right)$$

Figure 5.4.9 p 328

pseudo period $T = \frac{2\pi}{\omega_1}$

pseudo freq $f = \frac{\omega_1}{2\pi}$

pseudo ang. freq ω_1

Ce^{-pt} : " time varying amplitude

1. solution oscillates, but oscillations are damped exponentially
2. motion is slowed relative to no damping $(\omega_1 < \omega_0)$.

Example 2 page 328

add a little damping $(c=1)$ to example 1.

$$\tfrac{1}{2}x'' + x' + 50x = 0$$

$$\begin{cases} x'' + 2x' + 100x = 0 \\ x(0) = 1 \\ x'(0) = -5 \end{cases}$$

$r^2 + 2r + 100 = 0$ or $r = \dfrac{-2 \pm \sqrt{4-400}}{2}$

$(r+1)^2 + 99 = 0$

$r = -1 \pm i\sqrt{99}$ $= -1 \pm i\sqrt{99}$

$x(t) = e^{-t}\left(A\cos\sqrt{99}\,t + B\sin\sqrt{99}\,t\right)$ ← note, damping retards angular freq. $\omega_1 = \sqrt{99}$. ≈ 9.95

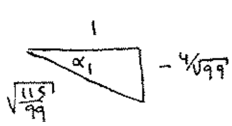
$x(0) = 1 = A$

$x'(0) = -5 = -A + \sqrt{99}\,B$

$B = -\frac{4}{\sqrt{99}}$

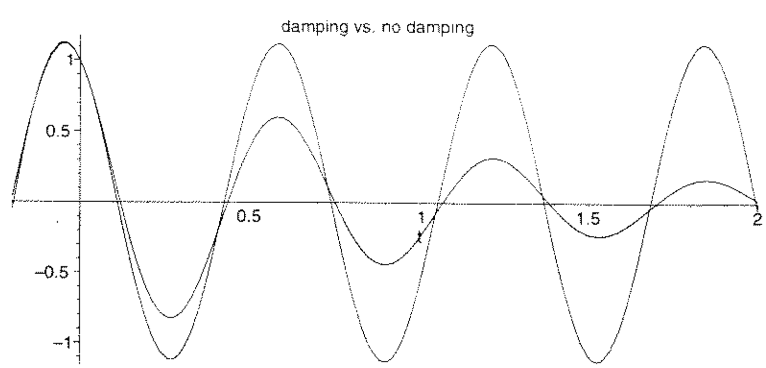
$$x(t) = e^{-t}\left[\cos(\sqrt{99}\,t) - \frac{4}{\sqrt{99}}\sin(\sqrt{99}\,t)\right]$$

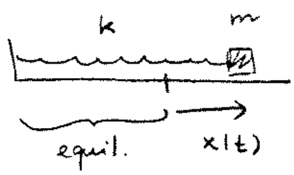
$$= e^{-t}\left[C_1\cos(\sqrt{99}\,t - \alpha_1)\right] \approx \boxed{1.078\,e^{-t}\cos(9.95t - -.38)}$$



α_1 larger

$\alpha_1 = \tan^{-1}\left(-4/\sqrt{99}\right)$





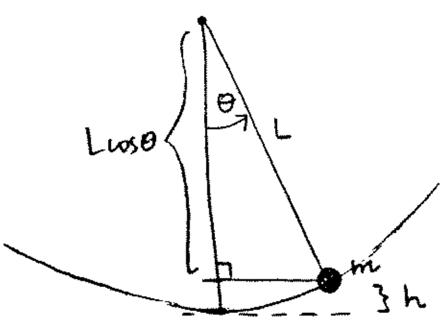
$$mx'' + cx' + kx = 0$$

c: damping coeff

k: Spring constant (Hooke's law)

units?

pendulum undamped:



E = total energy = KE + PE is constant

$$= \tfrac{1}{2}mv^2 + mgh$$

$s = L\theta$ (arclength from vertical)

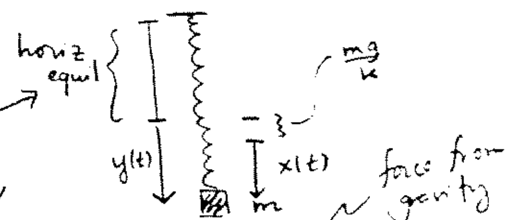
$$v = \frac{ds}{dt} = L\frac{d\theta}{dt}$$

$$h = L - L\cos\theta = L(1-\cos\theta)$$

so

$$E = \tfrac{1}{2}m(L\theta')^2 + mgL(1-\cos\theta)$$

take same spring/mass system & hang it vertically



$$my'' = mg - ky - cy'$$

$$my'' + cy' + ky = mg$$

let $y(t) = x(t) + \frac{mg}{k}$ $\quad (x = y - \frac{mg}{k})$

$$y' = x'$$

$$y'' = x''$$

$$mx'' + cx' + k\left(x + \frac{mg}{k}\right) = mg$$

$$x'' + cx' + kx = 0\,!$$

same!

$$0 = \frac{dE}{dt} = mL^2\theta'\theta'' + mgL\sin\theta\,\theta'$$

$$= mL\theta'\left[L\theta'' + g\sin\theta\right]$$

so $L\theta'' + g\sin\theta \equiv 0$

linearize! for small θ, $\sin\theta \approx \theta$

$$\boxed{L\theta'' + g\theta = 0}$$

or, with drag,

$$\boxed{L\theta'' + c\theta' + g\theta = 0}$$

Mathematically, this is the spring equation!