Math 2250-3
Wed 8 Dec.

Final exam: Tuesday Dec 14, 1-3 p.m.
here.

Review session: Fri Dec 10, 12:55-1:45
here

also, office hours Monday LCB 204
9:40-10:30
12:55-1:45

Exam is comprehensive:

15-20% { 1.1-1.5 — 1^{st} order DE's; 2.1-2.3 — applications, equilibria & stability
15-25% { 3.1-3.6 — matrix algebra & determinants; 4.1-4.5, 4.7 — vector space concepts
15-25% { 5.1-5.6 — linear DE's & spring applications
10-20% { 6.1 — eigenvalues & eigenvectors
15-25% { 7.1-7.4 — linear systems of DE's; tank & spring systems
15-20% { 10.1-10.4 — Laplace transforms; applications to DE's & DE systems

↑
possible percentages/topic

Analytic solution methods for DE's & systems of DE's
- 1^{st} order DE's: linear, separable
- higher order linear DE's: general soltn $= x_H + x_p$,
  - how to find x_H, x_p for constant coeff DE's,
  - use of Euler's formula for complex exponentials,
  - applications to mass-spring problems
- eigenvalue, eigenvector methods, + matrix computations to find $\vec{x}_p(t)$ for

$$\frac{d\vec{x}}{dt} = A\vec{x} + \vec{f}(t)$$

$$\frac{d^2\vec{x}}{dt^2} = A\vec{x} + \cos\omega t\,\vec{c}$$

  - applications to tank & mass-spring systems.
- Laplace transforms to solve linear DE's or DE systems. → you will get xeroxes of the book covers, i.e. basic Laplace tranforms and integrals.

Modeling: springs, spring systems
- compartmental analysis (tanks)
- population, velocity models, chptr 2

Geometric meaning for soltns to DE's & systems of DE's (and resulting "theorems").
- slope fields, phase portraits, equilibrium soltns, stability
- tangent vector fields (§7.1)
- existence/uniqueness for 1^{st} order DE's or DE systems, or n^{th} order DE's
- dimension of solution space for homogeneous DE's or DE systems

Auxillary tools
- linear algebra
  - new vocabulary, vector spaces, (in)dependence, span, dim, examples, etc.
  - solving linear systems, matrices, rref, matrix algebra, A^{-1}, dets, etc.
  - evals & evects
  - general sol'tn to $L(x)=b$ when L is a linear operator
- Laplace transform (def., methods e.g. partial fractions, translation, etc.)
  also, use in solving linear IVP's)

If we have time for some "fun":

Convolution

$$\boxed{f*g(t) := \int_0^t f(\tau)g(t-\tau)\,d\tau} \overset{\substack{\tilde{\tau}=t-\tau \\ d\tilde{\tau}=-d\tau}}{=} \int_0^t f(t-\tilde{\tau})g(\tilde{\tau})\,d\tilde{\tau} = g*f(t)$$

Laplace transform convolution

Theorem (see text page 604 for proof)

$$\boxed{\mathcal{L}\{f*g\}(s) = F(s)\,G(s)}$$

example $F(s)=G(s)=\frac{1}{s}=\mathcal{L}\{1\}$.

$$1*1(t) = \int_0^t 1\cdot 1\,d\tau = t$$

$$\mathcal{L}\{1*1\} = \mathcal{L}\{t\}(s) = \frac{1}{s^2} = \frac{1}{s}\cdot\frac{1}{s} \quad ✓$$

example

$$(\sin * \cos)(t) = \int_0^t (\sin\tau)\cos(t-\tau)\,d\tau$$

$$= \int_0^t \sin\tau\,(\cos t\cos\tau + \sin t\sin\tau)\,d\tau$$

$u=\sin\tau,\ du=\cos\tau\,d\tau$

$$= \cos t\int_0^t \sin\tau\cos\tau\,d\tau$$

$\sin^2\tau = \frac{1-\cos 2\tau}{2}$

$$+ \sin t\int_0^t \sin^2\tau\,d\tau$$

$$= \cos t\left(\tfrac{1}{2}\sin^2 t\right)$$

$$+ \sin t\left[\tfrac{1}{2}t - \frac{\overbrace{\sin 2t}^{2\sin t\cos t}}{4}\right]$$

$$= \tfrac{1}{2}t\sin t$$

so $\mathcal{L}\{\tfrac{1}{2}t\sin t\}(s) = \left(\frac{1}{1+s^2}\right)\left(\frac{s}{1+s^2}\right) = \frac{s}{(1+s^2)^2}$ ✓

$f(t)$	$F(s)$
1	$1/s$
t	$1/s^2$
t^n	$n!/s^{n+1}$ $\quad n\in\mathbb{N}$
e^{at}	$\frac{1}{s-a}$
$\cos kt$	$\frac{s}{s^2+k^2}$
$\sin kt$	$\frac{k}{s^2+k^2}$
$\cosh kt$	$\frac{s}{s^2-k^2}$
$\sinh kt$	$\frac{k}{s^2-k^2}$
$f'(t)$	$sF(s)-f(0)$
$f''(t)$	$s^2F(s)-sf(0)-f'(0)$
etc.	
$\int_0^t f(\tau)d\tau$	$\frac{1}{s}F(s)$
$t\,f(t)$	$-F'(s)$
$t^2 f(t)$	$F''(s)$
etc.	
$\frac{f(t)}{t}$	$\int_s^\infty F(\sigma)\,d\sigma$
$t\cos kt$	$(s^2-k^2)/(s^2+k^2)^2$
$\frac{1}{2k}t\sin kt$	$\frac{s}{(s^2+k^2)^2}$
$\frac{1}{2k^3}(\sin kt - kt\cos kt)$	$\frac{1}{(s^2+k^2)^2}$
$t\,e^{at}$	$\frac{1}{(s-a)^2}$
$u(t-a)$	e^{-as}/s
$u(t-a)f(t-a)$	$e^{-as}F(s)$
$e^{at}f(t)$	$F(s-a)$
$e^{at}\cos kt$	$\frac{s-a}{(s-a)^2+k^2}$
$e^{at}\sin kt$	$\frac{k}{(s-a)^2+k^2}$
$f*g(t)$	$F(s)\,G(s)$

analogons

analogons

today

<u>example</u> With convolution, can get soltn formulas in spring problems, for <u>any</u> forcing fcn.

$$\begin{cases} x'' + x = f(t) \\ x(0) = x_0 \\ x'(0) = v_0 \end{cases}$$

$\omega_0 = 1$, natural period is 2π

If $f(t)$ is periodic, we expect resonance if period is 2π. (and only if).

Are we right?

$$\mathcal{L} \downarrow$$

$$s^2 X(s) - s x_0 - v_0 + X(s) = F(s)$$

$$X(s)(s^2+1) = s x_0 + v_0 + F(s)$$

$$X(s) = \frac{s x_0 + v_0}{s^2+1} + \frac{1}{s^2+1} F(s)$$

so, $$x(t) = x_0 \cos t + v_0 \sin t + \underbrace{f * \sin(t)}_{\int_0^t \sin(t-\tau) f(\tau)\, d\tau}$$

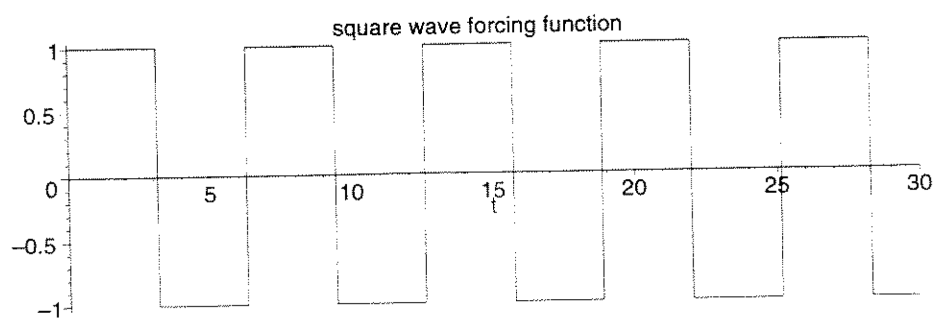
Now, let's play "Guess the resonance" game.

Math 2250-3
Wednesday December 8, 2004
Guess the resonance game

```
> with(plots):with(inttrans):
> f:=t->-1+2*sum((-1)^n*Heaviside(t-n*Pi),n=0..10);
```

$$f := t \to -1 + 2\left(\sum_{n=0}^{10} (-1)^n \text{Heaviside}(t - n\,\pi)\right)$$

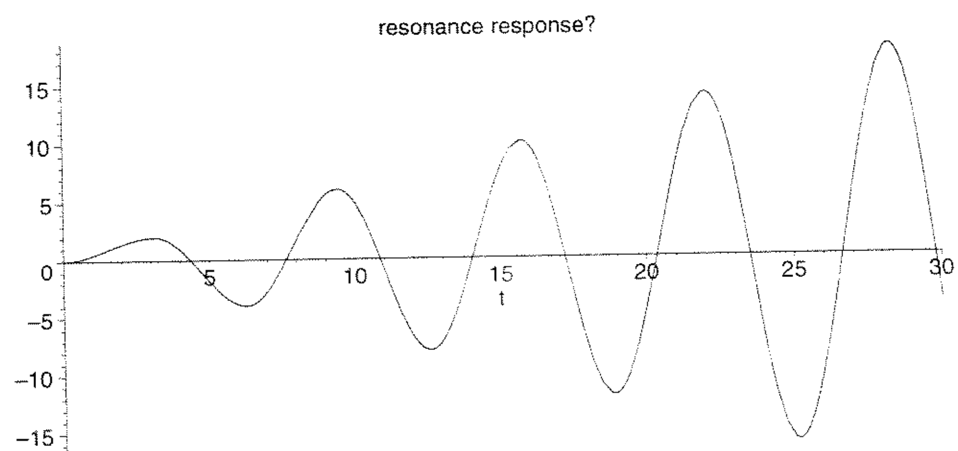
```
> plot(f(t),t=0..30,color=black,title=`square wave forcing
  function`);
```




```
> x:=t->int(sin(t-tau)*f(tau),tau=0..t);
```

$$x := t \to \int_0^t \sin(t - \tau)\,f(\tau)\,d\tau$$

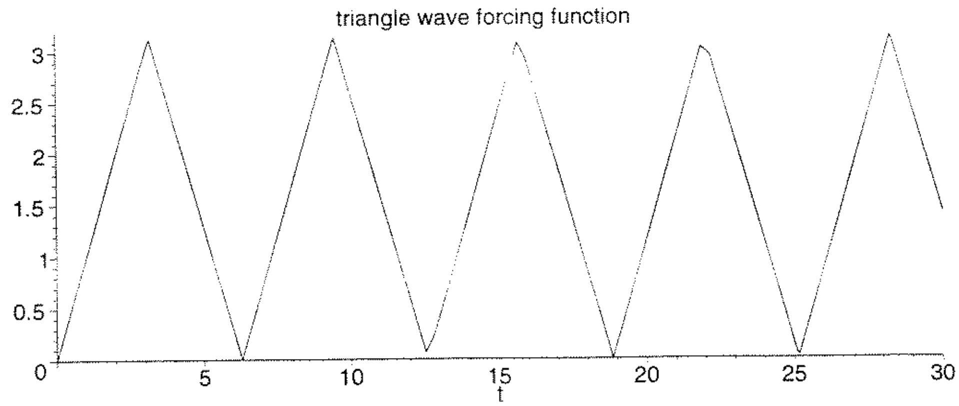
```
> plot(x(t),t=0..30,color=black,title=`resonance response?`);
```



```
> g:=t->int(f(u),u=0..t);
  #this should be a triangle wave...
```

$$g := t \to \int_0^t \mathrm{f}(u)\,du$$

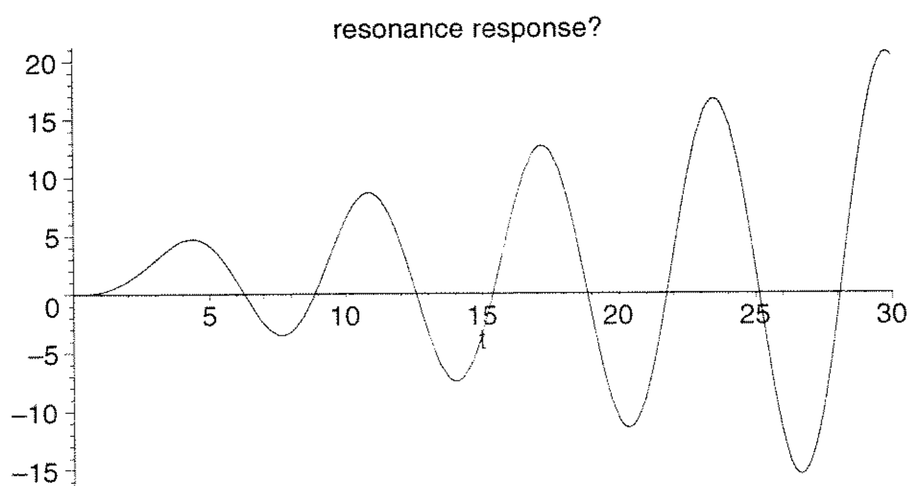
```
> plot(g(t),t=0..30,color=black, title='triangle wave forcing
  function');
```




```
> y:=t->int(sin(t-tau)*g(tau),tau=0..t);
```

$$y := t \to \int_0^t \sin(t-\tau)\,\mathrm{g}(\tau)\,d\tau$$

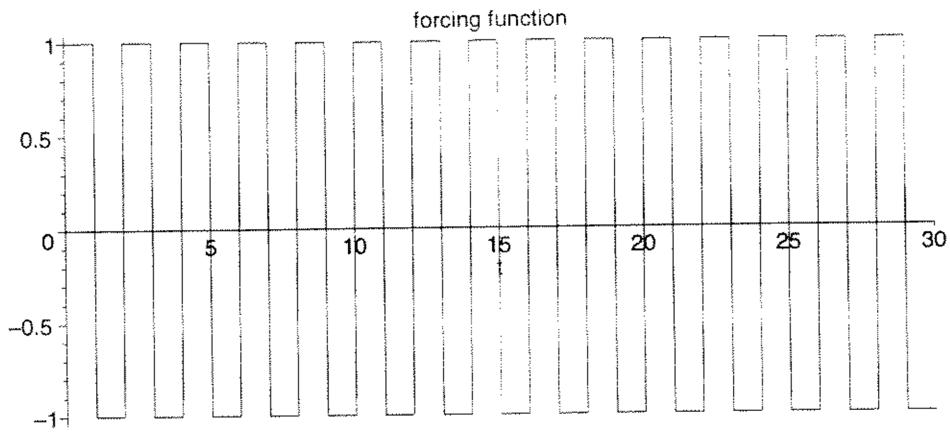
```
> plot(y(t),t=0..30,color=black, title='resonance response?');
```



```
> h:=t->-1+2*sum((-1)^n*Heaviside(t-n),n=0..30);
```

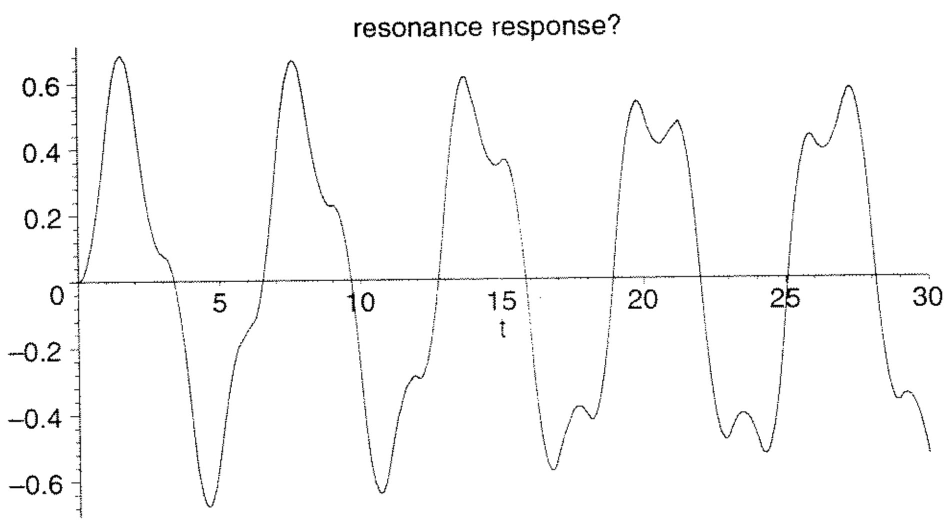
$$h := t \to -1 + 2\left(\sum_{n=0}^{30} (-1)^n \,\mathrm{Heaviside}(t-n)\right)$$

```
> plot(h(t),t=0..30,color=black, title='forcing function');
```




```
> z:=t->int(sin(t-tau)*h(tau),tau=0..t);
  plot(z(t),t=0..30,color=black,title='resonance response?');
```

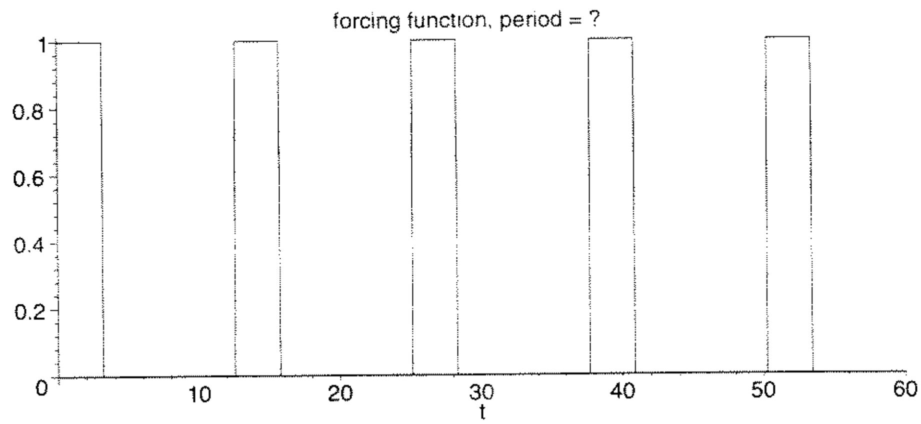
$$z := t \to \int_0^t \sin(t-\tau)\,\mathrm{h}(\tau)\,d\tau$$



```
> k:=t->sum(Heaviside(t-4*Pi*n)-Heaviside(t-4*Pi*n-Pi),
  n=0..5);
```

$$k := t \rightarrow \sum_{n=0}^{5} (\mathrm{Heaviside}(t - 4\,n\,\pi) - \mathrm{Heaviside}(t - 4\,n\,\pi - \pi))$$

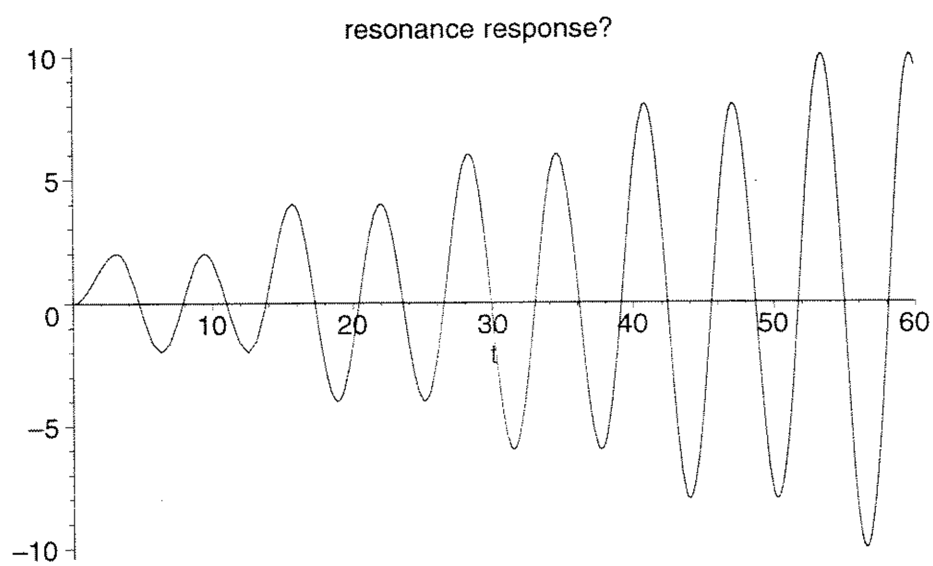
```
> plot(k(t),t=0..60,color=black,title=`forcing function, period =
  ?`);
>
```




```
> w:=t->int(sin(t-tau)*k(tau),tau=0..t);
```

$$w := t \rightarrow \int_0^t \sin(t - \tau)\,\mathrm{k}(\tau)\,d\tau$$

```
> plot(w(t),t=0..60,color=black,title=`resonance response?`);
```



Hey, what happened?