Math 2250
Monday 6 Dec.

Laplace transform cont'd.

to day:

$\mathcal{L}\{t\cos kt\}$:

$$\mathcal{L}\{\cos kt\}(s) = \frac{s}{s^2+k^2}$$

so $\mathcal{L}\{t\cos kt\} = -F'(s)$

$$= -\frac{1}{s^2+k^2} - s(-1)(s^2+k^2)^{-2}(2s)$$

$$= \frac{-s^2-k^2+2s^2}{(s^2+k^2)^2} = \frac{s^2-k^2}{(s^2+k^2)^2} \quad ✓$$

$\mathcal{L}\{t\sin kt\}$:

$$\mathcal{L}\{\sin kt\}(s) = \frac{k}{s^2+k^2}$$

so $\mathcal{L}\{t\sin kt\} = -F'(s)$

$$= -k(-1)(s^2+k^2)^{-2}2s$$

$$= \frac{+2ks}{(s^2+k^2)^2} \quad ✓$$

so $\mathcal{L}^{-1}\left(\frac{1}{(s^2+k^2)^2}\right) = \mathcal{L}^{-1}\left\{\left[\frac{s^2+k^2}{(s^2+k^2)^2} - \frac{(s^2-k^2)}{(s^2+k^2)^2}\right]\frac{1}{2k^2}\right\}(t)$

$$= \frac{1}{2k^2}\left(\frac{1}{k}\sin kt - t\cos kt\right) \quad ✓$$

(useful in resonance problems).

$x(t)$ $f(t)$	$X(s)$ $F(s)$
1	$1/s$
t	$1/s^2$
t^n	$n!/s^{n+1}$ $\quad n\in\mathbb{N}$
e^{at}	$1/(s-a)$
$\cos kt$	$s/(s^2+k^2)$
$\sin kt$	$k/(s^2+k^2)$
$\cosh kt$	$s/(s^2-k^2)$
$\sinh kt$	$k/(s^2-k^2)$
$f'(t)$	$sF(s)-f(0)$
$f''(t)$	$s^2F(s)-sf(0)-f'(0)$
etc	
$\int_0^t f(\tau)\,d\tau$	$\frac{F(s)}{s}$
$t f(t)$	$-F'(s)$
$t^2 f(t)$	$F''(s)$
etc	
$\frac{f(t)}{t}$	$\int_s^\infty F(\sigma)\,d\sigma$
$t\cos kt$	$(s^2-k^2)/(s^2+k^2)^2$
$\frac{1}{2k}t\sin kt$	$\frac{s}{(s^2+k^2)^2}$
$\frac{1}{2k^3}(\sin kt - kt\cos kt)$	$\frac{1}{(s^2+k^2)^2}$
te^{at}	$\frac{1}{(s-a)^2}$
$u(t-a)$	e^{-as}/s
$u(t-a)f(t-a)$	$e^{-as}F(s)$
$e^{at}f(t)$	$F(s-a)$
$e^{at}\cos kt$	$\frac{s-a}{(s-a)^2+k^2}$
$e^{at}\sin kt$	$\frac{k}{(s-a)^2+k^2}$
$f*g(t)$	$F(s)G(s)$

analogous

to day

(maybe Weds (for fun). §10.4)

Example p. 595

$$\begin{cases} x'' + \omega_0^2 x = F_0 \sin \omega t \\ x(0) = 0 \\ x'(0) = 0 \end{cases}$$

resonance (or not) revisited

$\mathcal{L}$:

$$s^2 X(s) - 0s - 0 + \omega_0^2 X(s) = F_0 \frac{\omega}{s^2+\omega^2}$$

$$X(s)\left[s^2+\omega_0^2\right] = F_0 \frac{\omega}{s^2+\omega^2}$$

$$X(s) = F_0\left(\frac{\omega}{s^2+\omega^2}\right)\left(\frac{1}{s^2+\omega_0^2}\right)$$

$\omega \neq \omega_0$

$$X(s) = F_0 \omega \left[\frac{1}{s^2+\omega^2} - \frac{1}{s^2+\omega_0^2}\right]\left[\frac{1}{\omega_0^2-\omega^2}\right]$$

so $$x(t) = \frac{F_0\omega}{\omega_0^2-\omega^2}\left[\frac{1}{\omega}\sin \omega t - \frac{1}{\omega_0}\sin \omega_0 t\right]$$

$\boxed{\text{much easier than method of undetermined coeffs!}}$

$\omega = \omega_0$ resonance

$$X(s) = \frac{F_0\omega_0}{(s^2+\omega_0^2)^2}$$

$x(t) =$ (from table!)

$$= \frac{F_0\omega_0}{2\omega_0^3}\left(\sin \omega_0 t - \omega_0 t \cos \omega_0 t\right)$$

$$x(t) = \frac{F_0}{2\omega_0^2}\left(\sin \omega_0 t - \omega_0 t \cos \omega_0 t\right)$$

resonance!

We could do the spring system problem (Friday notes),

or,

$$\begin{cases} y^{(4)} + 2y'' + y = 4te^t \\ y(0)=0 \\ y'(0)=0 \\ y''(0)=0 \\ y'''(0)=0 \end{cases}$$

(Example 6 p.596)

Should get

$$Y(s) = \frac{4}{(s-1)^2(s^2+1)^2}$$

$$= \frac{A}{s-1} + \frac{B}{(s-1)^2} + \frac{Cs+D}{s^2+1} + \frac{Es+F}{(s^2+1)^2}$$

so $y(t) = Ae^t + Bte^t + C\cos t + D\sin t + E\left(\frac{1}{2}t\sin t\right) + F\left(\frac{1}{2}(\sin t - t\cos t)\right)$

$$t \to \frac{1}{s^2}$$

$$te^{at} \to \frac{1}{(s-a)^2}$$

$$4 = A(s-1)(s^2+1)^2 + B(s^2+1)^2 + (Cs+D)(s^2+1)(s-1)^2 + (Es+F)(s-1)^2$$

$$4 = A(s-1)(s^4+2s^2+1) + B(s^4+2s^2+1) + (Cs+D)(s^2+1)(s^2-2s+1) + (Es+F)(s^2-2s+1)$$

Yipes!

```
> with(inttrans);
```

[*addtable, fourier, fouriercos, fouriersin, hankel, hilbert, invfourier, invhilbert, invlaplace, invmellin, laplace, mellin, savetable*]

```
> ?parfrac;
> ?invlaplace;
> F:=s->4/((s-1)^2*(s^2+1)^2);
```

$$F := s \to \frac{4}{(s-1)^2\,(s^2+1)^2}$$

```
> convert(F(s),parfrac,s);
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$$\frac{2\,s}{(s^2+1)^2} - \frac{2}{s-1} + \frac{1+2\,s}{s^2+1} + \frac{1}{(s-1)^2}$$

```
> invlaplace(F(s),s,t);
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$$(t+1)\sin(t) + 2\cos(t) + (-2+t)\,e^t$$