Math 2250-3
Fri Dec 3

More Laplace
15.2-15.3

I will be in
Marriott MMC PClab 1735 ("usual one")
today (Fri) 3:15-4:45 pm
& Saturday 2-4 pm
for MAPLE!

Recall

$$\underbrace{\mathcal{L}\{f(t)\}(s)}_{=F(s)} := \int_0^\infty e^{-st} f(t)\,dt$$

$\mathcal{L}$ is linear $\left(\mathcal{L}\{c_1 f_1(t) + c_2 f_2(t)\}(s) = c_1 F_1(s) + c_2 F_2(s)\right)$

$\mathcal{L}$ is invertible (inverse Laplace transform yields a function defined for $t \geq 0$)

e.g. $\mathcal{L}\{7 - 5\cos 3t + 2\sin 8t\}(s)$

$$= 7/s - \frac{5s}{s^2+9} + \frac{16}{s^2+64}$$

Fill in the table: (continue what we started ...)

① $g(t) = \int_0^t f(\tau)\,d\tau$

satisfies $g'(t) = f(t)$
$g(0) = 0$

so $\underbrace{\mathcal{L}\{g'(t)\}}_{=F(s)} = s\,G(s) - 0$

② $F(s) = \int_0^\infty e^{-st} f(t)\,dt$

$$F'(s) = \lim_{\Delta s \to 0} \frac{1}{\Delta s}\left(F(s+\Delta s) - F(s)\right)$$

$$= \lim_{\Delta s \to 0} \underbrace{\frac{1}{\Delta s} \int_0^\infty e^{-(s+\Delta s)t} f(t) - e^{-st} f(t)\,dt}_{\int_0^\infty f(t)\left[\frac{e^{-(s+\Delta s)t} - e^{-st}}{\Delta s}\right] dt}$$

$$\frac{d}{ds} e^{-st} = -t e^{-st}$$

$$= \int_0^\infty -t f(t) e^{-st}\,dt = \mathcal{L}\{-t f(t)\}(s)$$

	$f(t)$	$F(s)$
	1	$1/s$
	e^{at}	$1/(s-a)$
	$f'(t)$	$sF(s) - f(0)$
	$f''(t)$	$s^2F(s) - sf(0) - f'(0)$
	$f'''(t)$	$s^3F(s) - s^2f(0) - sf'(0) - f''(0)$
①	$\int_0^t f(\tau)\,d\tau$	$\frac{F(s)}{s}$
②	$t f(t)$	$-F'(s)$
	$t^2 f(t)$	$F''(s)$
	etc	
③	t	$\frac{1}{s^2}$
	t^2	$2/s^3$
	t^n	$n!/s^{n+1}$ n integer ≥ 0
	etc.	
4a	$\cos kt$	$\frac{s}{s^2+k^2}$
	$\sin kt$	$\frac{k}{s^2+k^2}$
4b	$\cosh kt$	$\frac{s}{s^2-k^2}$
	$\sinh kt$	$\frac{k}{s^2-k^2}$
5a	$u(t-a)$	e^{-as}/s
	$u(t-a)f(t-a)$	$e^{-as}F(s)$
5b	$e^{at}f(t)$	$F(s-a)$
	$e^{at}\cos kt$	$\frac{(s-a)}{(s-a)^2+k^2}$
	$e^{at}\sin kt$	$\frac{k}{(s-a)^2+k^2}$

(3) using (1): $\mathcal{L}(1) = 1/s$ did already

$$\int_0^t 1 \, \cancel{dt}\, d\tau = t$$

$$\mathcal{L}\left(\int_0^t 1 \, d\tau \right) = \frac{F(s)}{s} = \frac{1}{s^2}$$

$$\int_0^t \tau \, d\tau = \frac{1}{2} t^2$$

$$\mathcal{L}(\text{''}) = \frac{1}{s} \cdot \frac{1}{s^2} = \frac{1}{s^3}, \quad \text{so } \mathcal{L}\{t^2\}(s) = \frac{2}{s^3} \ldots$$

using (2): $\mathcal{L}\{t \cdot 1\} = -\frac{d}{ds}\left(\frac{1}{s}\right) = \frac{1}{s^2}$

$$\mathcal{L}\{t \cdot t\} = -\frac{d}{ds} s^{-2} = \frac{2}{s^3}$$

$$\mathcal{L}\{t \cdot t^2\} = -\frac{d}{ds} 2s^{-3} = \frac{3!}{s^4} \quad \text{etc.}$$

(4a) did Wed badly. better:

$$\mathcal{L}\{e^{ikt}\} = \frac{1}{s-ik} = \frac{s+ik}{s^2+k^2}$$

$$\parallel$$

$$\mathcal{L}\{\cos kt + i \sin kt\}(s)$$

(4b) $\cosh kt = \frac{1}{2}(e^{kt} + e^{-kt})$

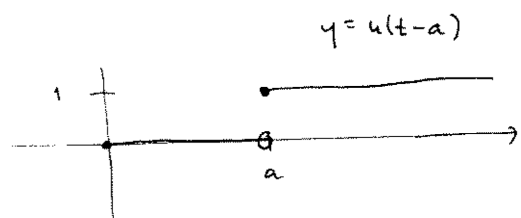
so $\mathcal{L}\{\cosh kt\}(s) = \frac{1}{2}\left(\frac{1}{s-k} + \frac{1}{s+k}\right) = \frac{s}{s^2-k^2}$

$\sinh kt = \frac{1}{2}(e^{kt} - e^{-kt})$

so $\mathcal{L}\{\sinh kt\}(s) = \frac{1}{2}\left(\frac{1}{s-k} - \frac{1}{s+k}\right) = \frac{k}{s^2-k^2}$

(5a) $u(t) := \begin{cases} 1 & t \geq 0 \\ 0 & t < 0 \end{cases}$ unit step fcn: turns things on and off (Maple: Heaviside fcn)

so $u(t-a) = \begin{cases} 1 & t-a \geq 0, \text{ i.e. } t \geq a \\ 0 & t < a \end{cases}$



$$\mathcal{L}\{u(t-a)\}(s) = \int_0^\infty e^{-st} u(t-a)\, dt$$

$$= \int_a^\infty e^{-st}\, dt = \left. \frac{e^{-st}}{-s} \right]_a^\infty = \frac{e^{-sa}}{s}$$

$$\mathcal{L}\{u(t-a) f(t-a)\}(s) = \int_a^\infty f(t-a) e^{-st}\, dt \qquad \tilde{t} = t-a, \quad d\tilde{t} = dt$$

$$= \int_0^\infty f(\tilde{t}) e^{-s(\tilde{t}+a)}\, d\tilde{t} = e^{-sa} F(s)!$$

(5b) $\mathcal{L}\{e^{at} f(t)\}(s) = \int_0^\infty e^{at} f(t) e^{-st}\, dt$

$$= \int_0^\infty f(t) e^{-(s-a)t}\, dt = F(s-a)$$

§10.3

Example 1
p. 591

$$\begin{cases} x'' + 6x' + 34 = 0 \\ x(0) = 3 \\ x'(0) = 1 \end{cases}$$

$$s^2 X(s) - 3s - 1 + 6(sX(s) - 3) + 34X(s) = 0$$

$$X(s)(s^2 + 6s + 34) = 3 \cdot s + 19$$

$$X(s) = \frac{3s + 19}{s^2 + 6s + 34} = \frac{3s + 19}{(s+3)^2 + 25}$$

$$= \frac{3(s+3) + 10}{(s+3)^2 + 25}$$

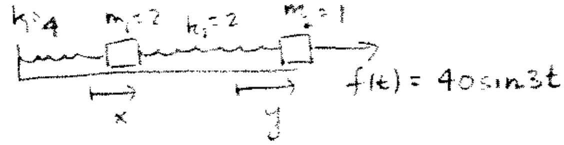
$$= 3\frac{s+3}{(s+3)^2 + 25} + \frac{10}{(s+3)^2 + 25}$$

$f(t) = \cos 5t$
$F(s) = \frac{s}{s^2+25}$
$F(s - -3)$
$e^{-3t} \cos 5t$

$f(t) = \sin 5t$
$F(s) = \frac{5}{s^2+25}$
$F(s - -3)$
$e^{-3t} \sin 5t$

$$\boxed{x(t) = 3e^{-3t}\cos 5t + 2e^{-3t}\sin 5t}$$

<u>Example</u> Spring system!



$$2x''=-6x+2y$$
$$y''=2x-2y+40\sin 3t$$
$$x(0)=x'(0)=y(0)=y'(0)=0$$

$\mathcal{L}$

$$\cancel{2}\left(s^2X(s)-sx(0)-0\right)=-\overset{3}{\cancel{6}}\,X(s)+\overset{1}{\cancel{2}}Y(s)$$
$$s^2Y(s)-sy(0)-0=2X(s)-2Y(s)+\frac{120}{s^2+9}$$

$$(s^2+3)X(s)-Y(s)=0$$
$$-2X(s)+(s^2+2)Y(s)=\frac{120}{s^2+9}$$

$$\begin{bmatrix} s^2+3 & -1 \\ -2 & s^2+2 \end{bmatrix}\begin{bmatrix} X(s) \\ Y(s) \end{bmatrix}=\begin{bmatrix} 0 \\ \frac{120}{s^2+9} \end{bmatrix}$$

so $$\begin{bmatrix} X(s) \\ Y(s) \end{bmatrix}=\frac{1}{(s^2+2)(s^2+3)-2}\begin{bmatrix} s^2+2 & 1 \\ 2 & s^2+3 \end{bmatrix}\begin{bmatrix} 0 \\ \frac{120}{s^2+9} \end{bmatrix}$$

$$s^4+5s^2+4$$
$$(s^2+4)(s^2+1)$$

$$X(s)=\frac{120}{(s^2+9)(s^2+4)(s^2+1)}=\frac{A}{s^2+9}+\frac{B}{s^2+4}+\frac{C}{s^2+1}=\frac{3}{s^2+9}-\frac{8}{s^2+4}+\frac{5}{s^2+1}$$

$$Y(s)=\frac{120(s^2+3)}{(s^2+9)(s^2+4)(s^2+1)}=\frac{D}{s^2+9}+\frac{E}{s^2+4}+\frac{F}{s^2+1}=\frac{-18}{s^2+9}+\frac{8}{s^2+4}+\frac{10}{s^2+1}$$

(similar)

parfrac!

$$120=A(s^2+4)(s^2+1)+B(s^2+9)(s^2+1)+C(s^2+9)(s^2+4)$$

$s=3i$: $120=A(-5)(-8)$; $A=3$

$s=2i$: $120=B(5)(-3)$; $B=-\frac{120}{15}=-8$

$s=i$: $120=C(8)(3)$ $C=\frac{120}{24}=5$

So $\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} \sin 3t - 4\sin 2t + 5\sin t \\ -6\sin 3t + 4\sin 2t + 10\sin t \end{bmatrix}$

$= \sin 3t \begin{bmatrix} 1 \\ -6 \end{bmatrix} + 4\sin 2t \begin{bmatrix} -1 \\ 1 \end{bmatrix} + 5\sin t \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

explain this interms of the spring problem

Maple:

Maple and Laplace
Math 2250-3

```
> with(inttrans);  #integral transform library
```

[*addtable, fourier, fouriercos, fouriersin, hankel, hilbert, invfourier, invhilbert, invlaplace, invmellin, laplace, mellin, savetable*]

```
> X:=s->120/((s^2+9)*(s^2+4)*(s^2+1));
  Y:=s->120*(s^2+3)/((s^2+9)*(s^2+4)*(s^2+1));
```

$$X := s \to \frac{120}{(s^2+9)(s^2+4)(s^2+1)}$$

$$Y := s \to \frac{120(s^2+3)}{(s^2+9)(s^2+4)(s^2+1)}$$

```
> invlaplace(X(s),s,t);
  invlaplace(Y(s),s,t);
```

$$-4\sin(2t) + 5\sin(t) + \sin(3t)$$

$$-6\sin(3t) + 4\sin(2t) + 10\sin(t)$$

```
> convert(X(s),parfrac,s);
  convert(Y(s),parfrac,s);
```

$$-\frac{8}{s^2+4} + \frac{5}{s^2+1} + \frac{3}{s^2+9}$$

$$\frac{8}{s^2+4} + \frac{10}{s^2+1} - \frac{18}{s^2+9}$$

```
> laplace(t*cos(k*t),t,s);
```

$$\frac{s^2-k^2}{(s^2+k^2)^2}$$