Math 2250-3
Wed Dec 1
Chapter 10

(Discuss Earthquake, pg 3 Monday)

Last HW!!
Due 12/8
10.1 3 7 8 13 20 21, 23 28
10.2 3, 4 5 6 14 19 20 28 31
10.3 3 7 8 17 20 27 31

Laplace transform: a way of transforming IVP's for linear systems of DE's directly into (solvable) algebra problems

$$\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt$$
$$\parallel$$
$$F(s)$$

$$\mathcal{L}\{1\}(s) = \int_0^\infty e^{-st}\,dt = \frac{e^{-st}}{-s}\Big]_0^\infty = \frac{1}{s} \qquad s>0$$

$$\mathcal{L}\{e^{at}\}(s) = \int_0^\infty e^{at} e^{-st}\,dt = \int_0^\infty e^{(a-s)t}\,dt = \frac{1}{a-s} e^{(a-s)t}\Big]_0^\infty = \frac{1}{s-a} \qquad s>a$$

$$\mathcal{L}\{\cos kt\}(s) = \int_0^\infty \tfrac{1}{2}(e^{ikt} + e^{-ikt}) e^{-st}\,dt = \frac{1}{2}\int_0^\infty e^{(ik-s)t} + e^{(-ik-s)t}\,dt$$
$$= \frac{1}{2}\left[\frac{e^{(ik-s)t}}{ik-s} + \frac{e^{(-ik-s)t}}{-ik-s}\right]_0^\infty \qquad s>0$$
$$= \frac{1}{2}\left(\frac{1}{s-ik} + \frac{1}{s+ik}\right)$$
$$= \frac{1}{2}\,\frac{s+ik+s-ik}{s^2+k^2} = \frac{s}{s^2+k^2}$$

$\mathcal{L}$ is linear:
$$\mathcal{L}\{c_1 f(t) + c_2 g(t)\}(s) = \int_0^\infty e^{-st}[c_1 f_1(t) + c_2 g(t)]\,dt = c_1 F(s) + c_2 G(s)$$

$$\mathcal{L}\{6 - 5e^{3t} + \cos 2t\}(s) = \frac{6}{s} - 5\frac{1}{s-3} + \frac{s}{s^2+4}.$$

$\mathcal{L}$ is also invertible [this is true but hard to show].
e.g. Use table on page 572 to find
$$\mathcal{L}^{-1}\left(\frac{7}{s^2} + \frac{4}{s^2+9}\right) \stackrel{?}{=}$$

Laplace transform and DE's :

$$\mathcal{L}\{f'(t)\}(s) = \int_0^\infty \underbrace{e^{-st}}_{v} \underbrace{f'(t)\,dt}_{du} = e^{-st} f(t)\Big]_{t=0}^{\infty} - \int_0^\infty -s e^{-st} f(t)\,dt$$

$u = f(t)$

$dv = -se^{-st}\,dt$

$$= -f(0) + sF(s)$$

if f grows at rate $e^{s't}$, $s'<s$

So,

$$\mathcal{L}\{f''(t)\}(s) = -f'(0) + s\,\mathcal{L}\{f'(t)\}(s)$$
$$= -f'(0) + s(-f(0) + sF(s))$$
$$= -f'(0) - sf(0) + s^2 F(s)$$

<u>**Example**</u> : Solve IVP

$$* \quad \begin{cases} x''(t) + 4x(t) = 10\cos 3t \\ x(0) = 2 \\ x'(0) = 1 \end{cases}$$

Write $\mathcal{L}\{x(t)\}(s) = X(s)$

If $*$ is true, then get equality when Laplace both sides

$$\mathcal{L}\{x''(t)\} + 4\mathcal{L}\{x(t)\} = 10\,\mathcal{L}\{\cos 3t\}$$

$$s^2X(s) - \overset{=1}{x'(0)} - s\overset{=2}{x(0)} + 4X(s) = 10\frac{s}{s^2+9}$$

$$X(s)(s^2+4) = 10\frac{s}{s^2+9} + 1 + 2s$$

$$X(s) = \frac{10s}{(s^2+4)(s^2+9)} + \frac{1+2s}{s^2+4}$$

$$= \underbrace{10s\left(\tfrac{1}{5}\right)}_{2s}\left(\frac{1}{s^2+4} - \frac{1}{s^2+9}\right) + \frac{1+2s}{s^2+4}$$

$$X(s) = -2\frac{s}{s^2+9} + \frac{1+4s}{s^2+4}$$

No need for x_p, x_H, $x_p + x_H$, IVP !

just a "teensy" bit of partial fractions

so $$\boxed{x(t) = -2\cos 3t + \tfrac{1}{2}\sin 2t + 4\cos 2t}$$

[table]