Math 2250-3
Notes for Separable Differential Equations - %1.4
Wednesday September 1, 2004

A differential equation

$$\frac{dy}{dx} = \mathrm{h}(x, y)$$

is called **separable** iff h(x,y) is a product of a function of x times a function of y,

$$\frac{dy}{dx} = \mathrm{g}(x)\,\phi(y)$$

This is equivalent to the DE

$$\frac{dy}{dx} = \frac{\mathrm{g}(x)}{\mathrm{f}(y)}$$

where f and phi are reciprocal functions.

How to solve:

The algorithm is very simple, but magic: treat dy/dx as a quotient of differntials (?!), and multiply through to rewrite the DE as

$$\mathrm{f}(y)\,dy = \mathrm{g}(x)\,dx$$

Then antidifferentiate the left side with respect to y and the right side with respect to x.

$$\int \mathrm{f}(y)\,dy = \int \mathrm{g}(x)\,dx + C$$

If F(y) and G(x) are antiderivatives of f(y) and g(x), respectively, then this is the solution

$$\mathrm{F}(y) = \mathrm{G}(x) + C$$

This equation defines y implicitly as a function of x. Sometimes you can use algebra to explicitly solve for y. The constant C can be adjusted to solve initial value problems.

Why the method works:

The use of differentials is disguising an application of the chain rule. Here is the explanation for the magic method: The differential equation

$$\frac{dy}{dx} = \frac{\mathrm{g}(x)}{\mathrm{f}(y)}$$

CAN be rewritten as

$$deqtn2 := \mathrm{f}(y)\left[\frac{dy}{dx}\right] = \mathrm{g}(x)$$

If y(x) is any solution to deqtn2, then the left side, namely

$$\mathrm{f}(\mathrm{y}(x))\left[\frac{dy}{dx}\right]$$

is the derivative with respect to x of

$$\mathrm{F}(\mathrm{y}(x))$$

whenever F(y) is an antiderivative of f(y) (with respect to y). This is just the chain rule! Thus if G(x) is any antiderivative of g(x) (w.r.t.x), we can legally antidifferentiate deqtn2 with respect to x (on both sides) to get

$$\mathrm{F}(\mathrm{y}(x)) = \mathrm{G}(x) + C$$

which is what we got by magic before!

Example 1 page 32: We wish to solve

$$\frac{dy}{dx} = -6\,x\,y$$

$$y(0) = 7$$

Work:

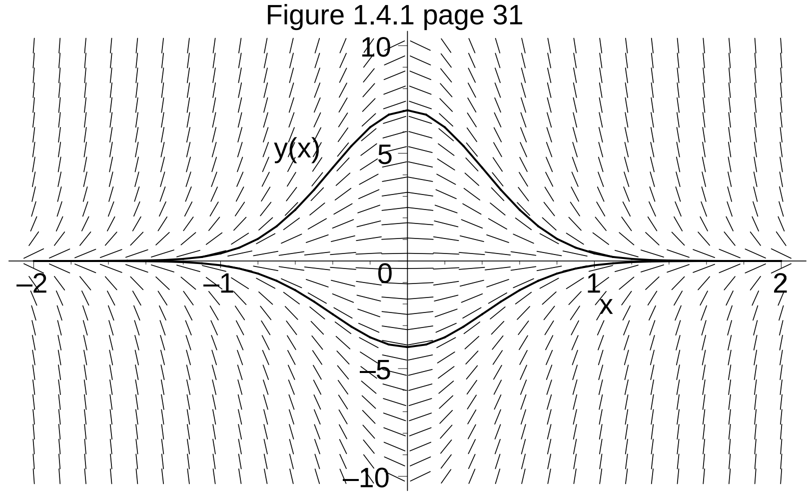
Notice, our method for the general solution doesn't actually give us the solution y(x)=0. Solutions which exist to separable DE's which are in addition to the ones we get are called **"singular solutions."**

slope field picture:

```
> restart:with(plots):with(DEtools):
> deqtn:=diff(y(x),x)=-6*x*y(x):  #this is example 2
  dsolve({deqtn,y(0)=7},y(x));  #Maple solution
  DEplot(deqtn,y(x),x=-2..2,{[y(0)=7],[y(0)=-4]},y=-10..10,arrows=li
  ne,  color=black,linecolor=black,dirgrid=[30,30],stepsize=.1,
   title=‘Figure 1.4.1 page 31‘); #slope field with two solution
  graphs
```

$$y(x) = 7\,\mathbf{e}^{(-3\,x^2)}$$

Figure 1.4.1 page 31



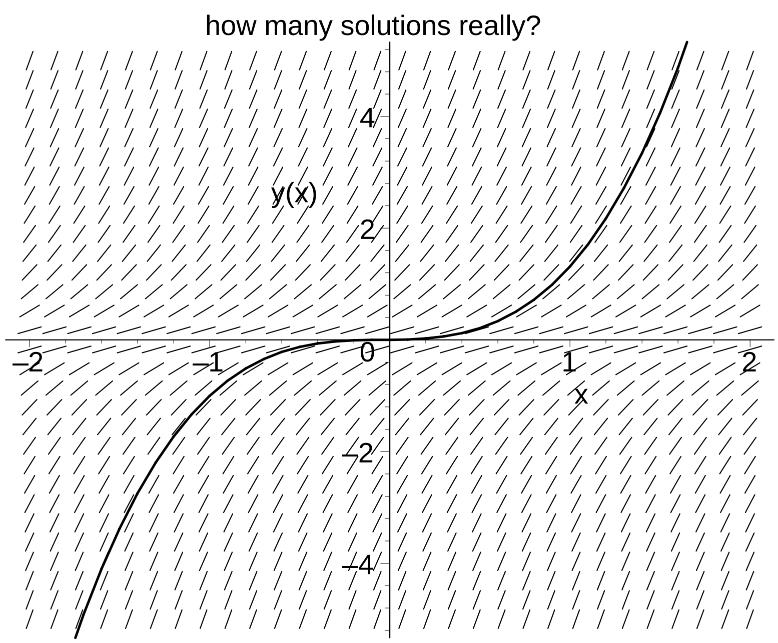
Example extra: (#29 page 28)

$$\frac{dy}{dx} = 3\,y^{(2/3)}$$

$$y(-1) = -1$$

Solution: (There is a twist to this problem that will let us discuss the existence-uniqueness theorem.)

```
> deqtn:=diff(y(x),x)=3*abs(y(x))^(2/3.0)*sign(y(x)):
  DEplot(deqtn,y(x),x=-2..2,{[y(-1)=-1]},y=-5..5,arrows=line,
  color=black,linecolor=black,dirgrid=[30,30],stepsize=.1,
   title=‘how many solutions really?‘); #Maple missed a few
```



Examples 2-3 page 32

```
> dy/dx=(4-2*x)/(3*y^2-5);
  y(1)=3;
```

$$\frac{dy}{dx} = \frac{4 - 2\,x}{3\,y^2 - 5}$$

$$y(1) = 3$$

Work:

```
> deqtn:=diff(y(x),x)=(4-2*x)/(3*y(x)^2-5):
  part1:=DEplot(deqtn,y(x),x=-5..7,y=-5..5,{[y(1)=3]},arrows=line,
     color=black,linecolor=black,dirgrid=[40,40],stepsize=.1,
     title='Part of Figure 1.4.2 page 32 '):
  with(plots):
  part2:=implicitplot(y^3-5*y=4*x-x^2+9,x=-5..7,y=-5..5,color=black)
  :
  display({part1,part2});
>
```

Part of Figure 1.4.2 page 32