Math 1210-001
Tuesday Mar 29
WEB L110

•   Complete Monday's notes -we still need to discuss "separable" differential equations.
•   Begin Chapter 4! Sections 4.1 and 4.2 are about "the definite integral". Section 4.1 uses the idea of "area under a graph" to motivate the general definition in 4.2. Along the way, we develop summation notation and formulas to help in our computations.

Space for warm-up differential equation problem:

Area: We all know the formulas for the area of basic regions in the plane:

Wisconsin Mathematics Formula Reference Sheet

Shape	Formulas for Area (A) and Circumference (C)
Triangle	$A = \frac{1}{2}bh = \frac{1}{2} \times$ base $\times$ height
Rectangle	$A = lw =$ length $\times$ width
Trapezoid	$A = \frac{1}{2}(b_1 + b_2)h = \frac{1}{2} \times$ sum of bases $\times$ height
Parallelogram	$A = bh =$ base $\times$ height
Circle	$A = \pi r^2 = \pi \times$ square of radius $C = 2\pi r = 2 \times \pi \times$ radius

Except for the area formula for the disk inside a circle, $A = \pi r^2$ in the chart above, the area formulas all follow from the formula for the areas of rectangles, and the fact that area is additive.

• Parallelograms and trapezoids can be decomposed, and the pieces reconfigured into rectangles.
• A triangle and a congruent copy of it can be configured into a parallelogram.

Archimedes used the letter π to denote the ratio of circumference of a circle to its diameter, i.e.
$\dfrac{C}{d} = \pi \Rightarrow C = 2\,\pi\,r$, where r is the radius. The method he used to deduce the area formula from that,

namely $A = \pi\,r^2$, is instructive and the ideas are relevant to what we will do in Chapter 4. The wikipedia page has more information, but we'll summarize.
https://en.wikipedia.org/wiki/Area_of_a_disk

Exercise 1) This diagram contains the idea of how to get the relationship between the area and circumference of a circle, using limits...discuss.
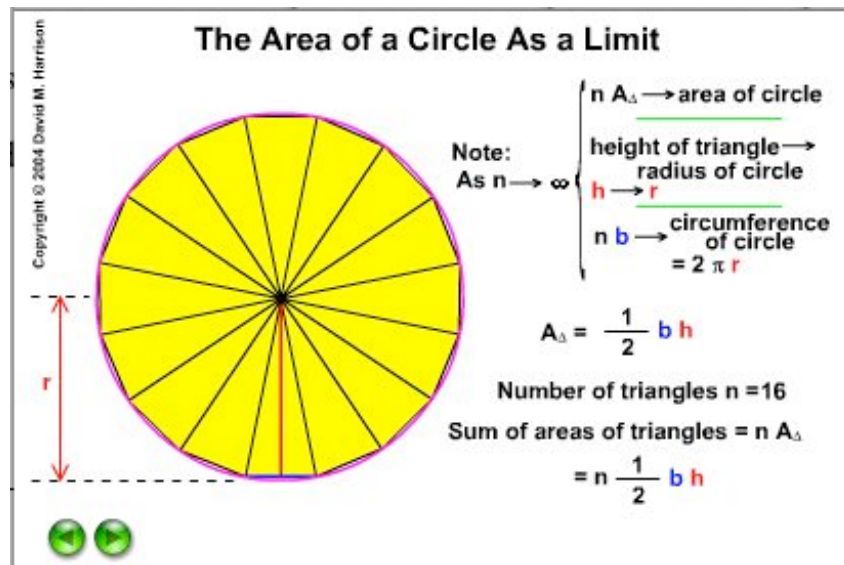

The idea is to approximate the circle area by the sum of areas of "n" inscribed congruent triangles, and then let $n \to \infty$. It's pretty clever.

One (of many) applications(s) of definite integrals has to do with area of regions in the plane, and it's perhaps the most intuitive application.

For example, suppose we want to calculate the area of the curved triangular region in the $x - y$ plane to the right of the $y - axis$, above the $x - axis$, to the left of the line $x = 1$, and under the parabola $y = x^2$.
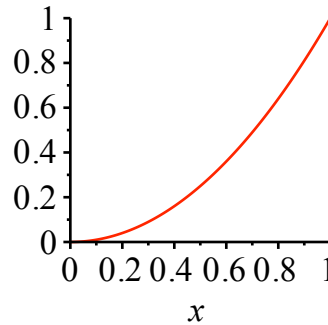
We can make under- and over-estimates for the area by "partioning" the x -interval $[0, 1]$ into subintervals, and computing the area of rectangles with bases on the subintervals, and heights related to the graph $y = x^2$. In the pictures below the interval was partitioned into 10 equal subintervals, each of width 0.1.
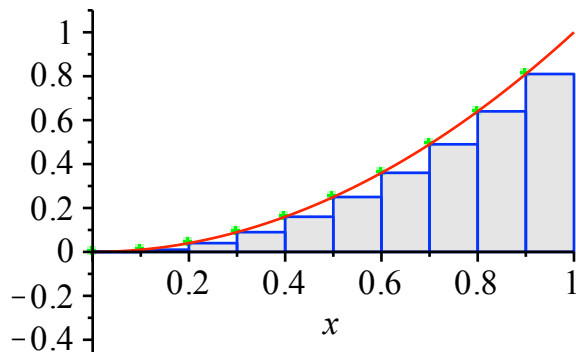
A left Riemann sum approximation of

$\int_0^1 f(x)\ dx$, where $f(x) = x^2$ and the

partition is uniform. The approximate value of the integral is 0.2850000000. Number of partitions used: 10.
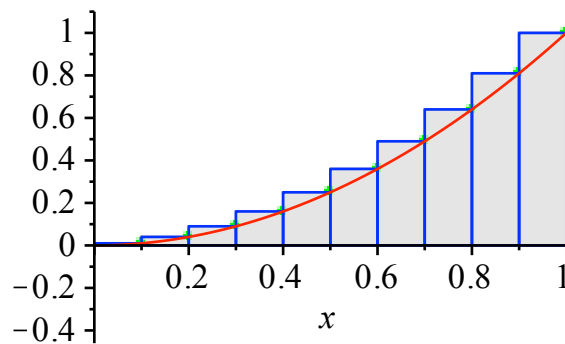
A right Riemann sum approximation of

$\int_0^1 f(x)\ dx$, where $f(x) = x^2$ and the

partition is uniform. The approximate value of the integral is 0.3850000000. Number of partitions used: 10.

Exercise 2) Write down the two formulas for the sums of the rectangle areas above

Check:

> $.1 \cdot (.1)^2 + .1 \cdot (.2)^2 + .1 \cdot (.3)^2 + .1 \cdot (.4)^2 + .1 \cdot (.5)^2 + .1 \cdot (.6)^2 + .1 \cdot (.7)^2 + .1 \cdot (.8)^2 + .1 \cdot (.9)^2;$

$$0.285 \tag{1}$$

What happens when we partition the interval $[0, 1]$ into 100 subintervals of length 0.01:

A left Riemann sum approximation of $\int_0^1 f(x)\ dx$, where $f(x) = x^2$ and the partition is uniform. The approximate value of the integral is 0.3283500000. Number of partitions used: 100.
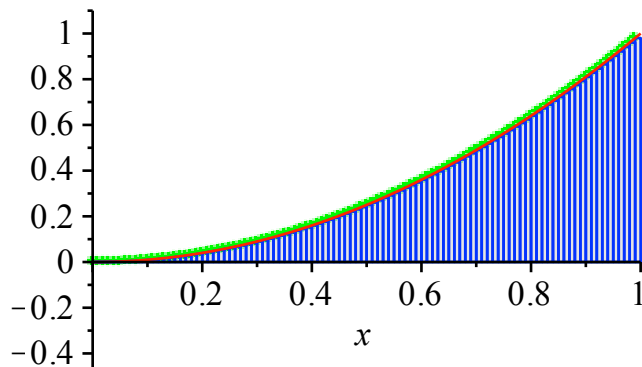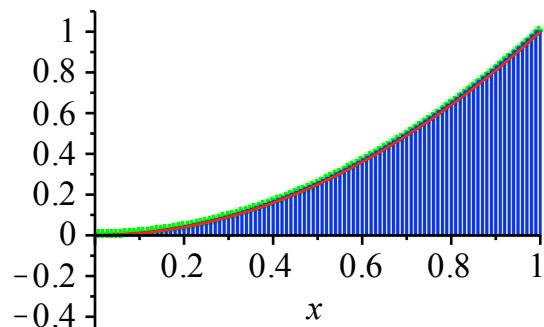
A right Riemann sum approximation of $\int_0^1 f(x)\ dx$, where $f(x) = x^2$ and the partition is uniform. The approximate value of the integral is 0.3383500000. Number of partitions used: 100.

There are notation shortcuts that make these sorts of summation problems easier to handle. This is the so-called "summation notation" and algebra:

Let $a_1, a_2, \ldots a_n$ be a list of n numbers. Then the sum

$$a_1 + a_2 + \ldots + a_n$$

is written in summation notation as

$$\sum_{i=1}^{n} a_i$$

"The sum of a_i as the index i ranges from 1 to n. The use of the letter "i" as the index is arbitrary - one could use "j,k" or other letters.

Example: When we subdivided the interval $[0, 1]$ into 10 equal subdivisions and used circumscribed rectangles, our area approximation was

$$.1 \cdot .1^2 + .1 \cdot .2^2 + .1 \cdot .3^2 + \ldots + .1 \cdot 1^2 = \sum_{i=1}^{10} .1 \cdot \left(\frac{i}{10} \right)^2.$$

Example:

$$\frac{1}{3} + \frac{1}{4} + \ldots + \frac{1}{20} = \sum_{j=3}^{20} \frac{1}{j}.$$

Exercise 3a) Write the following sum using summation notation

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots + \frac{1}{2^{20}}.$$

Exercise 3b) Expand the following sum into "long form", from the summation notation form:

$$\sum_{k=1}^{6} 4 + \frac{3}{k^2}.$$

Algebra for summation notation: In the computations below we assume c is a constant. If you write each summation out in long form, these become pretty clear:

$$\sum_{i=1}^{n} c = c + c + \ldots + c = nc$$

$$\sum_{i=1}^{n} c\, a_i = c \sum_{i=1}^{n} a_i$$

$$\sum_{i=1}^{n} \left(a_i + b_i \right) = \sum_{i=1}^{n} a_i + \sum_{i=1}^{n} b_i$$

$$\sum_{i=1}^{n} \left(a_i - b_i \right) = \sum_{i=1}^{n} a_i - \sum_{i=1}^{n} b_i$$

There are "magic formulas" for some special sums (see page 218):

$$\sum_{i=1}^{n} i = 1 + 2 + 3 + \ldots + n = \frac{n(n+1)}{2}$$

$$\sum_{i=1}^{n} i^2 = 1 + 2^2 + 3^2 + \ldots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

$$\sum_{i=1}^{n} i^3 = 1^3 + 2^3 + 3^3 \ldots + n^3 = \left[\frac{n(n+1)}{2} \right]^2$$

$$\sum_{i=1}^{n} i^4 = 1^4 + 2^4 + 3^4 + \ldots + n^4 = \frac{n(n+1)(2n+1)\left(3n^2 + 3n - 1\right)}{30}$$

....

<u>Punchline:</u>

Exact area of our parabolic triangular region: If we subdivide the interval $[0, 1]$ into n subintervals of length $\dfrac{1}{n}$ they are

$$\left[0, \frac{1}{n}\right], \left[\frac{1}{n}, \frac{2}{n}\right], \left[\frac{2}{n}, \frac{3}{n}\right], \dots \left[\frac{n-1}{n}, 1\right].$$

Since the rectangles are circumscribing the graph $y = x^2$ their heights are

$$\frac{1}{n^2}, \left(\frac{2}{n}\right)^2, \left(\frac{3}{n}\right)^2, \dots \left(\frac{n-1}{n}\right)^2, \left(\frac{n}{n}\right)^2$$

So the approximate area is

$$A_n = \sum_{i=1}^{n}\left(\frac{i}{n}\right)^2\left(\frac{1}{n}\right) = \sum_{i=1}^{n}\frac{i^2}{n^3}$$

$$= \frac{1}{n^3}\sum_{i=1}^{n}i^2$$

using the magic formula ...

$$= \frac{1}{n^3}\frac{(n)(n+1)(2n+1)}{6} = \frac{(n+1)\cdot(2n+1)}{6n^2}.$$

(When $n = 10$ we got $A_{10} = .385$, does this agree?)

The exact area is

$$\lim_{n\to\infty}\frac{(n+1)(2n+1)}{6n^2} = \frac{1}{3}.$$

$$= \lim_{n\to\infty}\frac{2n^2+3n+1}{6n^2} = \lim_{n\to\infty}\frac{2}{6} + \frac{3}{6n} + \frac{1}{6n^2} = \frac{1}{3}.$$