Math 1210-001
Wednesday Mar 23
WEB L112

• Which WebWork or lab problem(s) shall we look at, at the end of class? (Your current 3.4-3.5 WebWork assignment due date has been extended until tonight at midnight.

• The quiz for this Friday will be an optimization problem of difficulty level that one might expect on an exam (as opposed to some of the lab/WebWork problems you've been working on which would not be exam-appropriate).

• 3.8 Antidifferentiation (We do not need to finish theses notes today, we just need to get a good start.)

<u>Definition</u>: $F(x)$ is an <u>antiderivative</u> of $f(x)$ on the interval I means

$$D_x F(x) = f(x) \quad \text{for all } x \text{ in } I.$$

<u>Exercise 1</u>) (We started this exercise on Monday.) Find all antiderivatives of $f(x) = x^2$. Hint: first find one antiderivative, then find some more, then show there are no more. The last step will use the Mean Value Theorem.

Theorem A (the "+C" theorem)

a) If $F(x)$ is an antiderivative to $f(x)$ on the interval I, and if C is a constant, then $F(x) + C$ is also an antiderivative to $f(x)$.

b) If F is an antiderivative to f on the interval I and if G is any other antiderivative to f, then $G = F + C$ for some constant C. In other words, all antiderivatives of $f(x)$ are of the form $F(x) + C$, where $F(x)$ is any single antiderivative.

The proof works just like our discussion of Exercise 1.

a) $\mathrm{D}_x(F(x) + C) = \mathrm{D}_x(F(x)) + \mathrm{D}_x C = f(x) + 0 = f(x)$.

b) $\mathrm{D}_x(G(x) - F(x)) = \mathrm{D}_x(G(x)) - \mathrm{D}_x(F(x)) = f(x) - f(x) = 0$ for all x in the interval. So for the function $H(x) = G(x) - F(x)$, and any fixed a in I

$$\frac{H(x) - H(a)}{x - a} = H'(c)$$

by the Mean Value Theorem, and since $H'(c) = 0$, $H(x) = H(a)$, so

$$G(x) - F(x) = H(a)$$
$$G(x) = F(x) + H(a) = F(x) + C.$$

Notation. For reasons which will become more clear as we use it, and also in Chapter 4, we use Leibniz' notation for antiderivatives. Namely the symbol

$$\int f(x)\,\mathrm{d}x$$

is read as "the antiderivative of $f(x)$ with respect to x". The symbol $\int$ looks like a stretched out S and is also called an "integral sign". So the complete expression for antiderivative is also sometimes read as "the integral of $f(x)$ with respect to x".

Example: $\int x^2\,\mathrm{d}x = \frac{x^3}{3} + C$

Exercise 2) Find the following

2a)

$$\int x^3 \, dx$$

2b)

$$\int t^2 + 7\,t + 6 \, dt$$

Exercise 3) Use our differentiation rules in reverse, to find the following antiderivatives

3a) $\displaystyle \int 3\,x^7 - \frac{4}{x^2} + \frac{5}{\sqrt{x}} \, dx =$

3b) $\displaystyle \int 4\cdot\cos(t) + 8\cdot\sec^2(t) + 3\csc(t)\tan(t) \, dt$

3c) $\displaystyle \int u^n \, du$ (as long as $n \neq -1$)

3d) $\int f'(x)\, \mathrm{d}x$ for any differentiable function $f(x)$

3e) $\mathrm{D}_x\left(\int f(x)\, \mathrm{d}x\right)$ for any function $f(x)$ which has antiderivatives

Remark: 3d,e show that differentiation and antidifferentiation are almost inverse operations. 3d says that antidifferentiation undoes differentiation up to additive constants; 3e says that differentiation undoes antidifferentiation.

Each differentiation rule can be read in reverse to get an antidifferentiation rule. We saw how that works with the power rule and the various trigonometry derivative rules, in the previous exercise.

Theorem B

(i) $\int k f(x)\, \mathrm{d}x = k\int f(x)\, \mathrm{d}x$ constant multiple rule in reverse

(ii) $\int f(x) + g(x)\, \mathrm{d}x = \int f(x)\, \mathrm{d}x + \int g(x)\, \mathrm{d}x$ sum rule in reverse

(iii) $\int f'(g(x))g'(x)\, \mathrm{d}x = f(g(x)) + C$ chain rule in reverse

(iv) $\int g(x)^n\, g'(x)\mathrm{d}x = \frac{1}{n+1} g(x)^{n+1} + C$ generalized power rule, special case of (iii).

(v) $\int f(x)g'(x)\, \mathrm{d}x = f(x)g(x) - \int f'(x)g(x)\, \mathrm{d}x$ product rule in reverse (called integration by parts -see Math 1220)

Exercise 4) Find the following antiderivatives

4a) $\int (x^4 + 3x)^{30} (4 \cdot x^3 + 3) \, dx =$

4b) Differential substitution (called "the method of substitution") makes it smoother to do the chain rule in reverse. In the previous example,

$$\int (x^4 + 3x)^{30} (4 \cdot x^3 + 3) \, dx$$

substitute for the "inner" function

$$u = x^4 + 3x$$
$$du = (4x^3 + 3) dx$$

and redo the problem that way.

4c) Try differential substitution to solve

$$\int \frac{x}{\sqrt{x^2 + 4}} \, dx.$$

Test your answer by differentiating it.

4d) Compute

$$\int \sin^2(2x)\cos(2x) \, dx.$$