Math 1210-001
Tuesday Mar 22
WEB L110

• WebWork problems?

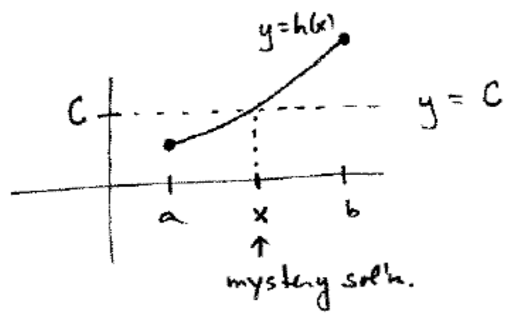
• 3.7 solving equations numerically
(This means, using algorithms to get approximate numerical solutions, especially in cases where the decimal approximations are important and might not be obtainable directly via algebra.)

<u>Setup:</u> Suppose $h(x)$ is a function, C is constant, and we want a solution to the equation

$$h(x) = C.$$

Here's one way to proceed:
Suppose we also know

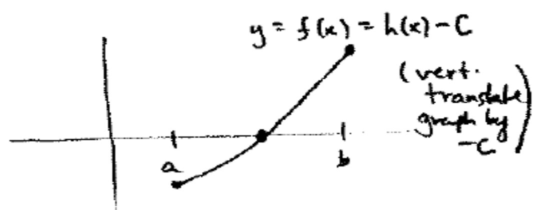
- $h(x)$ is continuous on some interval $[a, b]$
- $h(a) < C$ and $h(b) > C$
  (or alternately, $h(a) > C, h(b) < C$.)



<u>Step 1</u>: This is the same as finding a <u>root</u> of the equation

$$f(x) = 0$$

where $f(x) := h(x) - C$.

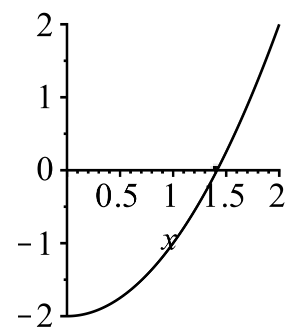


<u>Step 2:</u> Use the "bisection method" to narrow in on a solution to $f(x) = 0$, i.e. to $h(x) = C$ in the original setup. At each stage you cut the interval in half, and keep the subinterval for which the sign of f changes between the two endpoints (assuming you don't actually get lucky and stumble onto an exact root). More precisely, use $x < X$ to denote the sub-interval endpoints, beginning with $x = a, X = b$. Let $z = \dfrac{x + X}{2}$ be the midpoint. If $f(x), f(z)$ have opposite signs, keep x, but update X to be z. Conversely, if $f(z), f(X)$ have opposite signs, keep X but update x to be z. (There's a tiny chance that $f(z) = 0$, in which case you've found an exact root.)

If you repeat the bisection algorithm n times (without stumbling on an exact root), the current x and X will be a distance $\frac{b-a}{2^n}$ apart, and $f(x), f(X)$ will have opposite signs. So, as $n \to \infty$, x and X will converge to a number c for which the values of f on one side are positive, and on the other side are negative, so since f is continuous, we will have $f(c) = 0$. If we stop after a finite number of steps we can determine c to whatever accuracy we wish.

Remark: Notice that this algorithm proves the so-called Intermediate Value Theorem: Let $h(x)$ be continuous on the interval $[a, b]$, and let C be a ("intermediate") value between $h(a)$ and $h(b)$. Then there exists a c in $[a, b]$ with $h(c) = C$.

Exercise 1) Use the bisection method to approximate $\sqrt{2}$.
Solution: We are searching for the positive x that solves $x^2 = 2$. So, we want a positive root of the equation $x^2 - 2 = 0$. We'll choose the initial interval $[a, b] = [1, 2]$ since $1^2 - 2 = -1 < 0$, $2^2 - 2 = 2 > 0$. Here's a graph of $y = x^2 - 2$ so you can see the approximate location of the root:



Complete the first four rows of the following bisection method table by hand or calculator, then observe the continuation via Maple, on the next page.

$$f(x) = x^2 - 2$$

x	X	$f(x)$	$f(X)$	$z = \frac{x+X}{2}$	$f(z)$
1	2	-1	2	1.5	0.25
1	1.5				

```
> f := t→t^2 − 2 :
  x := 1 : X := 2 :
  for i from 1 to 10 do
      z := .5·(x + X) :
      if f(x)·f(z) > 0 then x := z end if:
      if f(x)·f(z) < 0 then X := z : end if:
```

```
print(x, X);
end do:
```

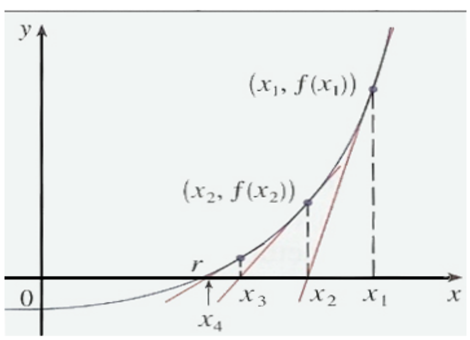
1, 1.5
1.25, 1.5
1.375, 1.5
1.375, 1.4375
1.40625, 1.4375
1.40625, 1.421875
1.4140625, 1.421875
1.4140625, 1.41796875
1.4140625, 1.416015625
1.4140625, 1.415039062 **(1)**

So, we conclude $1.414 < \sqrt{2} < 1.4151$. We could do more steps if we wanted more accuracy. It is a lot of work for a lot of accuracy however, and so there is a much quicker way that often works to find roots, namely Newton's method:

Newton's Method to find a solution x to the equation

$$f(x) = 0$$

for functions f that are differentiable (whereas the bisection method only requires functions to be continuous). First, find a "good" first guess for the root, x_0, based on information you may know about f. Then, get successive guesses x_{n+1} from x_n by choosing the x-intercept from the tangent line to the graph of f at the point $(x_n, f(x_n))$. Here's the geometric picture, where the root we're searching for is labeled as "r".



algebraic details: tangent line at $(x_n, f(x_n))$ has slope $f'(x_n)$ so its equation in point slope form is

$$y - f(x_n) = f'(x_n)(x - x_n).$$

If we set $y = 0$ to find the $x - intercept$ we solve for x in

$$-f(x_n) = f'(x_n)(x - x_n) \Rightarrow -\frac{f(x_n)}{f'(x_n)} = x - x_n \Rightarrow x_n - \frac{f(x_n)}{f'(x_n)} = x.$$

So the iteration formula is

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} := g(x_n).$$

Exercise 2) Use Newton's method to re-estimate $\sqrt{2}$. Use $f(x) = x^2 - 2$, $x_0 = 1$.

2a) Show that the iteration function $g(x) = x - \frac{f(x)}{f'(x)}$ simplifies to $g(x) = \frac{x}{2} + \frac{1}{x}$.

2b) Fill in x_1, x_2, x_3 in the table:

x_0	1
x_1	
x_2	
x_3	

2c) Compare to numerical work using software. Notice how fast the convergence is.

```
> Digits := 40 : #why not use 40 significant digits?
  f := t→t^2 − 2 :
  g := t→ t/2 + 1/t :
  x := 1.0 : #initial guess
  for i from 1 to 5 do
  x := g(x) :
  print(g(x));
  end do:
```

1.416666666666666666666666666666666666667
1.414215686274509803921568627450980392157
1.414213562374689910626295578890134910117
1.414213562373095048801689623502530243615
1.414213562373095048801688724209698078570 **(2)**

```
> sqrt(2.0); #Maple's value for square root of 2, to 40 digits
```

1.414213562373095048801688724209698078570 **(3)**

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>
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