Math 1210-001
Wednesday Jan 27
WEB L112

2.1-2.2 Introduction to Derivatives, continued.

• Yesterday, for a graph $y = f(x) = x^2$, we used limits of slopes of secant lines, to compute tangent line slopes to the graph. At a point on the graph with input coordinate "x", we computed

$$m_{\tan} = \lim m_{\sec} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

For $f(x) = x^2$ we saw that the slope function was given by $m_{\tan}(x) = 2\,x$.

• Finish <u>Exercise 2</u> from yesterday, which recast essentially an equivalent discussion into different words: average rates of change of a function (equivalent to secant line slopes of its graph) and instantaneous rates of change for a function (equivalent to tangent line slopes of its graph), in that case for a height function of a vertically-moving object,

$$g(t) = -16\,t^2 + 64\,t \; ft$$

at time t sec.

Motivated by these two discussions we make the following definitions:

1) For a function $f(x)$ defined on an open interval containing c, the <u>derivative of f at c</u> is denoted by $f'(c)$ ("f prime at c") and is defined as

$$f'(c) := \lim_{h \to 0} \frac{f(c+h) - f(c)}{h},$$

provided the limit exists. If the limit above exists, we say that f is differentiable at c. If the limit does not exist, we say that f is not differentiable there.

2) The <u>difference quotients</u>

$$\frac{f(c+h) - f(c)}{h}$$

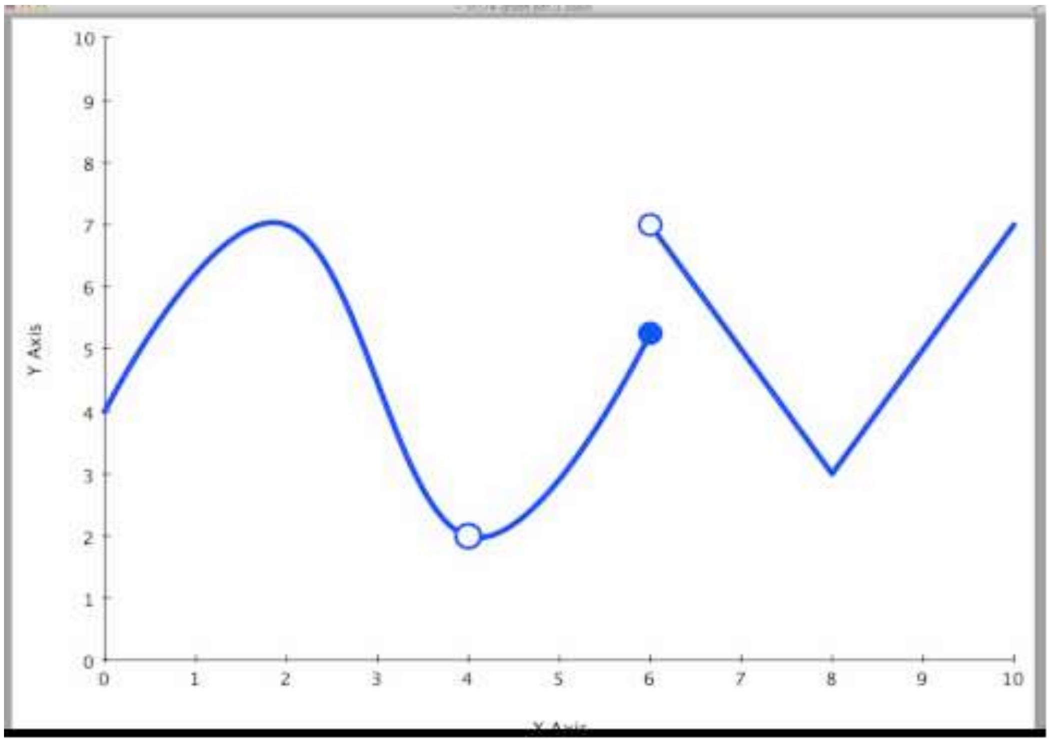
represent the slopes of secant lines on the graph of f. Equivalently, they are the average rates of change of f on the intervals $c \le x \le c + h$. The units for the difference quotients are $\frac{\textit{output units}}{\textit{input units}}$, e.g. $\frac{ft}{\text{sec}}$.
These are also the units for $f'(c)$.

We will now study derivatives more systematically.

Exercise 1 Let $f(x) = mx + b$ be any function whose graph is a line. Compute $f'(c)$ using the limit definition of derivative. Does your answer make sense?

Exercise 2 Let $f(x) = |x|$ be the absolute value function. Is it differentiable for all x? What happens at $x = 0$?

Exercise 3) Consider the graph of a mystery function, shown below. Discuss continuity and differentiability for all x values $0 < x < 10$. (I obtained the picture from the URL http://i.ytimg.com/vi/_vimObBaJxI/0.jpg.)



Theorem: If the function f is differentiable at $x = c$, then it is also continuous at $x = c$.
reason: Let f be differentiable at $x = c$. Then

$$\lim_{x \to c} f(x)$$
$$= \lim_{x \to c} f(c) + (f(x) - f(c))$$
$$= f(c) + \lim_{x \to c} (f(x) - f(c))$$
$$= f(c) + \lim_{x \to c} \frac{f(x) - f(c)}{x - c} \cdot (x - c)$$
$$= f(c) + \lim_{x \to c} \frac{f(x) - f(c)}{x - c} \lim_{x \to c} (x - c)$$
$$f(c) + f'(c) \cdot 0 = f(c).$$

□

Note: We used an alternate way of writing the limit definition of derivative $f'(c)$:

$$f'(c) = \lim_{h \to 0} \frac{f(c+h) - f(c)}{h} = \lim_{x \to c} \frac{f(x) - f(c)}{x - c}.$$

The connection between the two limits is the relationship $x = c + h$, and $h \to 0$ is equivalent to $x \to c$.

Exercise 4) The preceding fact means that whereever a function is differentiable, it is also continuous. Is the converse true? i.e. is it true that wherever a function is continuous, it is also differentiable?

Shortcuts for computing derivatives:

Theorem: Let f and g be differentiable at c. Let k be a constant Then $f + g$ and $k f$ are also differentiable at c and

$$(f+g)'(c) = f'(c) + g'(c)$$
$$(kf)'(c) = kf'(c).$$

Why?

Exercise 5) Yesterday we showed that for $f(x) = x^2, f'(x) = 2x$. In Exercise 1 we showed that for $g(x) = mx + b, g'(x) = m$. What is the derivative $h'(x)$ for

$$h(x) = 27x^2 - 56x + 133\ ?$$

Exercise 6) Use the limit definition of derivative, to find $f'(1)$ for $f(x) = \sqrt{2-x}$.