

Math 1210-001  
Wednesday Jan 20  
WEB L110

Begin

- Section 1.6 Continuity of functions
- Section 1.4 Limits of trigonometric functions

1.6 The easiest limits to compute are those for which we can just use the function value at the limiting input point to compute the limit, i.e. when

$$\lim_{x \rightarrow c} f(x) = f(c).$$

(Here  $c, f(c)$  are (finite) real numbers.) Geometrically, when the limit can be computed in this way nothing too wild is happening to the function  $f$  at the point  $c$ : for example, the one-sided limits at  $x = c$  must both exist and be equal, and they must agree with the function value there. This condition is special, and has a special name:

Def Let  $f$  be defined on an open interval containing  $c$ . We say that  $f$  is continuous at  $c$  if

$$\lim_{x \rightarrow c} f(x) = f(c).$$

Note: This requirement actually requires three things:

- 1)  $\lim_{x \rightarrow c} f(x)$  exists
- 2)  $x = c$  is in the domain of  $f$ , i.e.  $f(c)$  exists.
- 3)  $\lim_{x \rightarrow c} f(x) = f(c)$ .

Exercise 1: Consider the rational function

$$f(x) = \frac{x^2 - 2x - 3}{x - 3}.$$

1a) Explain why  $f$  is continuous at each  $c \neq 3$ . What theorem are you using?

1b) Note that  $x = 3$  is not in the domain of  $f$ . Is it possible to define  $f(3)$  in such a way so as to make  $f$  continuous there?

1c) Using the  $f(3)$  you found in 1b sketch the graph of  $f$  and discuss, in the context of the concept of continuity.

Definition If  $f$  can be defined or redefined at  $x = c$  so that  $f$  becomes continuous there, we say that  $f$  has a removable discontinuity at  $x = c$ . (That's what happened in Exercise 1.)

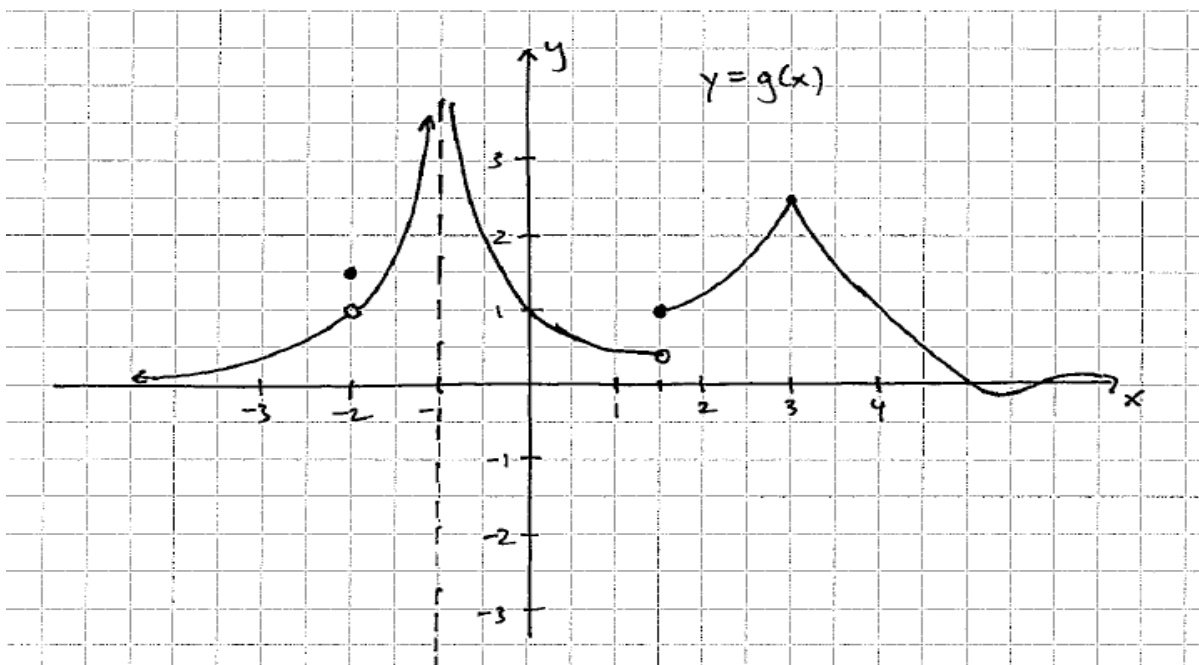
Exercise 2: Discuss the continuity of the function  $g$  graphed below, specifically at

2a)  $x = 0$

2b)  $x = -2$

2c)  $x = 1.5$

2d)  $x = 3$



Because of the limit theorems we've discussed, we know lots of functions that are continuous at all or almost all values  $c$ . Let's discuss the following theorems and why they're true:

Theorem A (Continuity of polynomials and rational functions) A polynomial function is continuous at every real number  $c$ . A rational function is continuous at every real number  $c$  in its domain, i.e. everywhere except where its denominator is zero.)

Theorem B (Continuity of absolute value and  $n^{\text{th}}$  root functions) The absolute value function is continuous at every real number  $c$ ; if  $n$  is odd, the  $n^{\text{th}}$  root function is continuous at every real number  $c$ . If  $n$  is even, the  $n^{\text{th}}$  root function is continuous at every positive real number  $c$ .

Theorem C (Continuity under function operations) If  $f, g$  are continuous at  $c$  then the following functions are also continuous:  $kf, f + g, f - g, f \cdot g, \frac{f}{g}$  (provided  $g(c) \neq 0$ );  $f^n, \sqrt[n]{f}$  (provided  $f(c) > 0$  if  $n$  is even).

Exercise 3) At what values of  $c$  is the following function continuous?

$$f(x) = \frac{x^3 - 5}{\sqrt{x} + \sqrt[5]{x}}.$$

Theorem D (Continuity of trigonometric functions) The  $\sin(x)$  and  $\cos(x)$  functions are continuous at every real number  $c$ . The functions  $\tan(x)$ ,  $\cot(x)$ ,  $\sec(x)$ ,  $\csc(x)$  are continuous at every real number  $c$  in their domains.

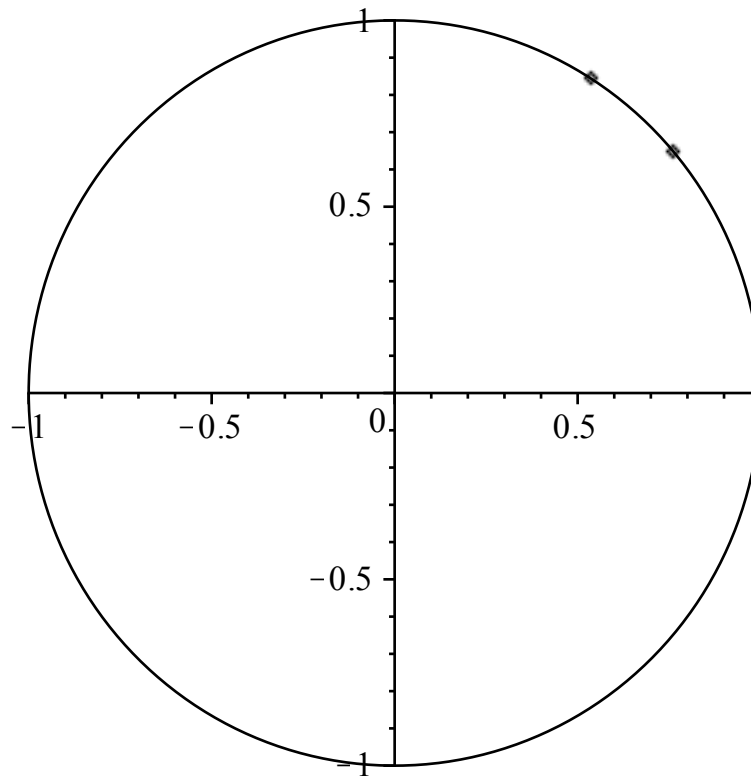
Reason: We can deduce the theorem once we show that  $\sin(x)$  and  $\cos(x)$  are continuous functions, since the other trig functions are obtained from those and the function 1, by taking various quotients...so Theorem C (for quotients) applies.

To show that  $\sin(t)$  and  $\cos(t)$  are continuous, we must show that

$$\lim_{t \rightarrow c} \cos(t) = \cos(c)$$

$$\lim_{t \rightarrow c} \sin(t) = \sin(c).$$

We'll use the following diagram to carefully convince ourselves, and recall a bit of trigonometry basics:



Exercise 4) Determine all points of discontinuity for the function  $f(x) = \frac{2}{1 + \sin(x)}$ .

Theorem E (Composite limit theorem) If  $\lim_{x \rightarrow c} g(x) = L$  and if  $f$  is continuous at  $L$ , then

$$\lim_{x \rightarrow c} f(g(x)) = f\left(\lim_{x \rightarrow c} g(x)\right) = f(L).$$

In particular, if  $g$  is continuous at  $c$  and  $f$  is continuous at  $g(c)$ , then the composition  $f \circ g$  is continuous at  $c$ .

Exercise 5) Find

$$\lim_{x \rightarrow \frac{\pi}{4}} \sin(3x)$$
$$\lim_{x \rightarrow \frac{\pi}{4}} (\sin(3x))^3.$$

There is one more limit theorem from section 1.3 that is very important. It's related to the ideas we used to show continuity for the *sin* and *cos* functions:

Theorem D section 1.3 (Squeeze Theorem): Let  $f, g, h$  be functions satisfying

$$f(x) \leq g(x) \leq h(x)$$

for all  $x$  near  $c$ , except possibly at  $c$ . If

$$\lim_{x \rightarrow c} f(x) = L = \lim_{x \rightarrow c} h(x)$$

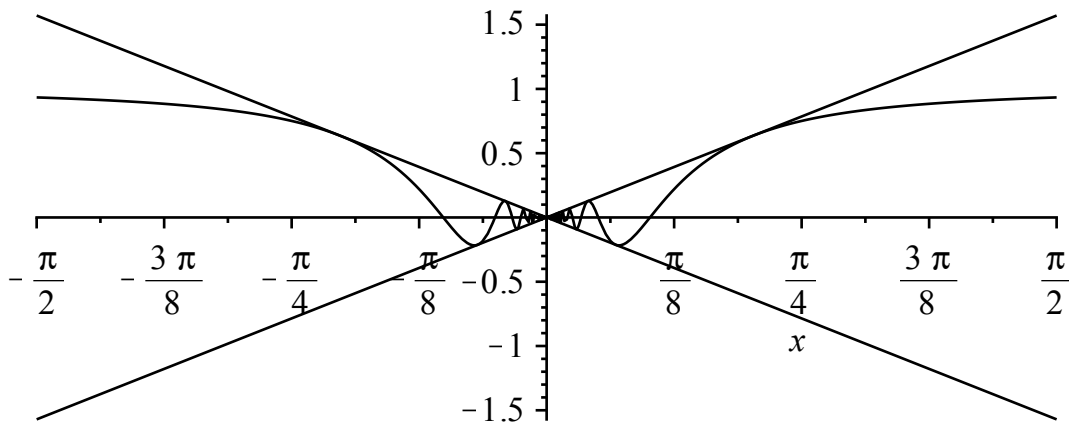
then  $\lim_{x \rightarrow c} g(x)$  also exists, and equals  $L$  too.

Exercise 6) Use the squeeze theorem to show

$$\lim_{x \rightarrow 0} x \sin\left(\frac{1}{x}\right) = 0$$

even though  $\lim_{x \rightarrow 0} \sin\left(\frac{1}{x}\right)$  does not exist. Hint, consider the helpful functions  $f(x) = -|x|$ ,  $h(x) = |x|$ .

this graph also shows what's going on:



to be continued ...