Math 1210-001
Wednesday Jan 13
WEB L112

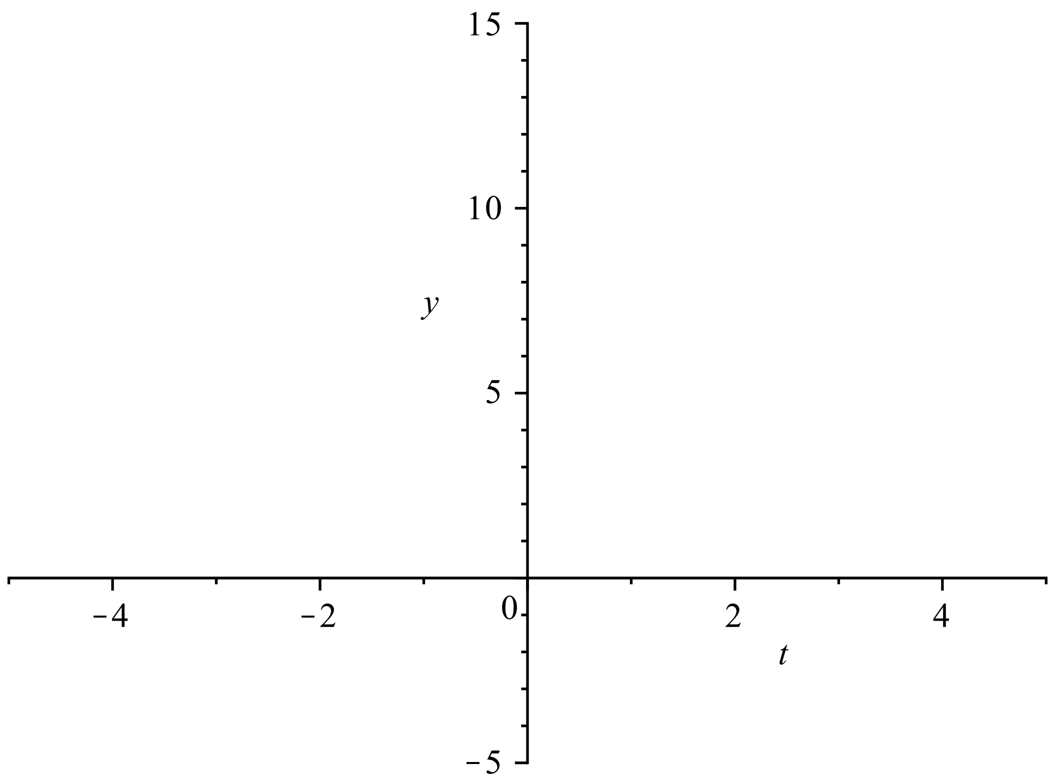
Finish 1.1, Introduction to Limits, and begin 1.3, Limit Theorems.

Yesterday we discussed limits and one-sided limits. What do each of these three statements below mean?

$$\lim_{x \to c} f(x) = L \qquad \lim_{x \to c^+} g(x) = M \qquad \lim_{x \to c^-} h(x) = N$$

Exercise 1) This was the last exercise on yesterday's notes:
1a) Sketch the graph of the piecewise-defined function

$$g(t) = \begin{cases} -2, & -\infty < t < -3 \\ 1 & t = 3 \\ -t^2 + 9, & -3 < t < 3 \\ 2t - 6, & t \geq 3 \end{cases}$$



b) Discuss existence of limits and one-sided limits at
(i) $t = -3$
(ii) $t = 1$
(iii) $t = 3$

Theorem A from Section 1.1: $\lim_{x \to c} f(x)$ exists and equals L if and only if both one sided limits $\lim_{x \to c^+} f(x)$ $\lim_{x \to c^-} f(x)$ exist and also equal L.

Example: Consider the absolute value function

$$|x| = \begin{cases} -x & x \le 0 \\ x & x \ge 0 \end{cases}$$

Example: Reconsider the piecewise-defined function on the previous page, at e.g. $t = 3$, $t = -3$.

1.3 Limit Theorems

There are shortcuts to computing limits. These Theorems are discussed in section 1.3. Let's discuss why the various claims collected below are true. This discussion will hint at the precise definition in section 1.2, which we do not focus on in this course. The underlying ideas will have to do with estimating error, and this is an important thing to be able to do in science and engineering, so it's worth taking some time for the discussion.

Theorem A from 1.3: Let n be a positive integer, k a constant, f, g be functions that have limits at c. Then

1. $\lim_{x \to c} k = k$
2. $\lim_{x \to c} x = c$
3. $\lim_{x \to c} kf(x) = k\lim_{x \to c} f(x)$
4. $\lim_{x \to c} [f(x) + g(x)] = \lim_{x \to c} f(x) + \lim_{x \to c} g(x)$
5. $\lim_{x \to c} [f(x) - g(x)] = \lim_{x \to c} f(x) - \lim_{x \to c} g(x)$
6. $\lim_{x \to c} [f(x) \cdot g(x)] = \lim_{x \to c} f(x) \cdot \lim_{x \to c} g(x)$
7. $\lim_{x \to c} \left[\dfrac{f(x)}{g(x)} \right] = \dfrac{\left(\lim_{x \to c} f(x)\right)}{\lim_{x \to c} g(x)}$ provided $\lim_{x \to c} g(x) \neq 0$
8. $\lim_{x \to c} [f(x)]^n = \left[\lim_{x \to c} f(x)\right]^n$
9. $\lim_{x \to c} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x \to c} f(x)}$.

Exercise 2 Compute $\lim_{x \to -2} 5x^2 - 16x$ step by step, using Limit Theorem A and labeling each step with the number of the fact from Theorem A that you used.

Exercise 3 Find $\lim_{t \to 4} \frac{5t}{\sqrt{t^2 + 9}}$

Using the same reasoning, we see that if $p(x)$ is a polynomial function,

$$p(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0$$

then we can compute $\lim_{x \to c} p(x)$, by just substituting in $x = c$, i.e.

$$\lim_{x \to c} p(x) = p(c).$$

And if $q(x)$ is another rational function

$$q(x) = b_n x^n + b_{n-1} x^{n-1} + \ldots + b_1 x + b_0$$

is another rational function with $q(c) \neq 0$ then for the rational function $r(x) = \frac{p(x)}{q(x)}$,

$$\lim_{x \to c} \frac{p(x)}{q(x)} = \frac{p(c)}{q(c)}$$

In other words,

Theorem B (Substitution Theorem) If f is a polynomial function or a rational function with $f(c)$ defined,

$$\lim_{x \to c} f(x) = f(c).$$

Exercise 4 Find

$$\lim_{x \to 2} \frac{7x^3 - 10x^2 + 2}{5x + 3}.$$

Exercise 5 Find

$$\lim_{t \to 4} \frac{t - 4}{\sqrt{t} - 2}.$$

(Note, this is like problems we were doing yesterday, in which substitution doesn't immediately work.)

Our reasoning above also explains:

Theorem C If $f(x) = g(x)$ for all x in an open interval containing the number c except possibly at c itself, and if $\lim_{x \to c} g(x)$ exists, then $\lim_{x \to c} f(x)$ also exists, and the two limits are equal,

$$\lim_{x \to c} g(x) = \lim_{x \to c} f(x).$$