Monday supplementary notes...These are definitions and facts from last Wednesday's notes:

Definition Let f be defined on an interval I (open, closed, or neither). Then

(i) f is increasing on I means that for all points $x_1 < x_2$ in I, $f(x_1) < f(x_2)$. (In other words, the values of f increase as the input values x increase.)
(ii) f is decreasing on I means that for all points $x_1 < x_2$ in I, $f(x_1) > f(x_2)$. (In other words, the values of f decrease as the input values x increase.)
(iii) f is strictly monotone on I means f is either increasing or decreasing on I.

The following theorem makes intuitive sense, based on our understanding that $f'(x)$ is the slope function for the graph of f. We'll discuss a more formal proof later in the chapter:

Theorem A (Monotonicity Theorem) Let f be continuous on the interval I and differentiable at every interior point (i.e. non endpoint) of I. Then
(i) If $f'(x) > 0$ for all x interior to I then f is increasing on I.
(ii) If $f'(x) < 0$ for all x interior to I then f is decreasing on I.

Definition Let f be differentiable on an open interval I. Then
(i) f (or the graph of f) is concave up means that the slope function f' is increasing on I.
(ii) f (or the graph of f) is concave down means that the slope function f' is decreasing on I.

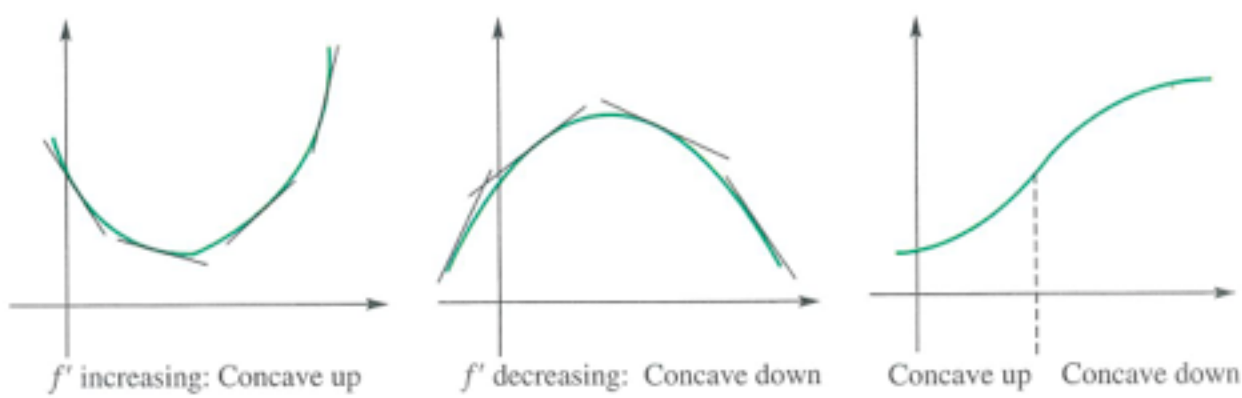


Figure 7

Applying Theorem A - but to the slope function f' and its derivative f'' we obtain

Theorem B (Concavity Theorem) Let f be twice differentiable on the open interval I. Then
(i) If $f''(x) > 0$ for all x in I, then f is concave up on I.
(ii) If $f''(x) < 0$ for all x in I, then f is concave down on I.

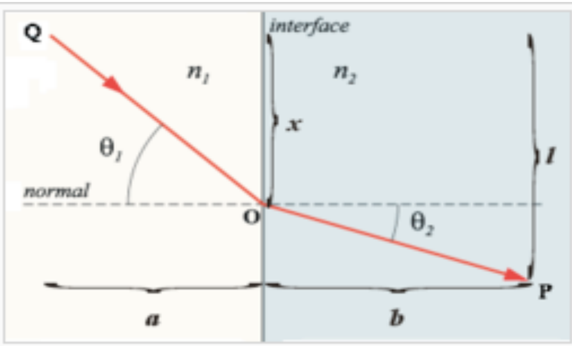
Application example (either today or tomorrow):

Snell's Law for how light refracts when it moves from one medium to another (for example from water to air, or vise verse):

Let the speed of light in medium one be v_1, and let it be v_2 in medium two. Assume the two media are separated by a plane (as in figure). Let θ_1 and θ_2 be the angles the light paths make with the perpendicular to the interface. Then Snell's Law says

$$\frac{\sin(\theta_1)}{v_1} = \frac{\sin(\theta_2)}{v_2} \quad \text{or equivalently} \quad \frac{\sin(\theta_1)}{\sin(\theta_2)} = \frac{v_1}{v_2} .$$

Snell's law is a consequence of Calculus (stationary point and first derivative test), and Fermat's Principle - which says that light follows the paths that minimize the total time it takes for the light to get from point Q to point P in all media.



Light from medium 1, point Q, enters medium 2, refraction occurs, and reaches point P finally.

Here's the explanation, which we will discuss and enlarge upon.

Let T be the time required for the light to travel from point Q to point P.

$$T = \frac{\sqrt{x^2 + a^2}}{v_1} + \frac{\sqrt{b^2 + (l-x)^2}}{v_2}$$

$$\frac{dT}{dx} = \frac{x}{v_1\sqrt{x^2+a^2}} + \frac{-(l-x)}{v_2\sqrt{(l-x)^2+b^2}} = 0 \text{ (stationary point)}$$

Note that $\dfrac{x}{\sqrt{x^2+a^2}} = \sin\theta_1$

$$\frac{l-x}{\sqrt{(l-x)^2+b^2}} = \sin\theta_2$$

$$\frac{dT}{dx} = \frac{\sin\theta_1}{v_1} - \frac{\sin\theta_2}{v_2} = 0$$

https://en.wikipedia.org/wiki/Snell%27s_law