Math 1210-001
Tuesday Feb 2
WEB L110

2.3: shortcut rules for taking derivatives.

On Monday we understood and worked with these derivative rules:

$D_x(1) = 0$
$D_x(x) = 1$
$D_x\left(x^2\right) = 2x$
$D_x\left(x^n\right) = n\,x^{n-1},\ n \in \mathbb{N}.$ (power rule; includes the three special cases above.)
$D_x(f(x) + g(x)) = D_x(f(x)) + D_x(g(x))$ (sum rule)
$D_x(kf(x)) = k\,D_x(f(x))$ if k is a constant. (constant multiple rule)

Towards the end of class we saw the product rule and the quotient rule, for differentiating products and quotients of functions. They will be our topic for today:

$D_x(f(x)g(x)) = f'(x)g(x) + f(x)g'(x)$ (product rule)

$$D_x\left(\frac{f(x)}{g(x)}\right) = \frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2}.$$ (quotient rule)

(Note that the numerator of the quotient rule almost looks like the product rule - except that one term has a positive sign and the other one has a minus sign. Some people remember the quotient rule with a jingle: "The derivative of high over low, is low d-high minus high d-low, over low squared".)

Exercise 1) Compute

$$D_x\left(\left(x^3 - 4x\right)\left(x^4 + 3\right)\right).$$

two ways: Once with the product rule, and once by first expanding the product function into a degree 7 polynomial. Verify that your answers agree.

Exercise 2) Use the quotient rule to find

$$D_t\left(\frac{t^2 + 3\,t}{t - 3}\right)$$

Exercise 3)

a) Use the quotient rule to find

$$D_x\left(x^{-3}\right) = D_x\left(\frac{1}{x^3}\right)$$

b) Use the quotient rule to find

$$D_x\left(x^{-k}\right)$$

when k is a counting number.

c) Conclude that the power rule, $D_x\left(x^n\right) = n\,x^{n-1}$ holds for all integer values n, $n \in \mathbb{Z}$.

d) Find $D_x\left(4\,x^{10} + \frac{5}{x^2} - \frac{6}{x} + \frac{27}{x^5}\right)$.

Why are the product rule and the quotient rule true?

<u>step 0</u>) We will need to use the fact that differentiable functions are continuous. This was in last Wednesday's notes. We skipped the proof at the time, although we discussed examples.

<u>Theorem</u>: If the function f is differentiable at c, then it is also continuous at c.
reason:

$$\lim_{x\to c} f(x)$$
$$= \lim_{x\to c} f(c) + (f(x) - f(c))$$
$$= f(c) + \lim_{x\to c} (f(x) - f(c))$$
$$= f(c) + \lim_{x\to c} \frac{f(x) - f(c)}{x - c} \cdot (x - c)$$
$$= f(c) + \lim_{x\to c} \frac{f(x) - f(c)}{x - c} \lim_{x\to c} (x - c)$$
$$f(c) + f'(c) \cdot 0 = f(c).$$

Thus, we showed that because $f'(c)$, exists, $\lim_{x\to c} f(x) = f(c)$, i.e. f is continuous at c.

$\square$

<u>Note</u>: We used an alternate way of writing the limit definition of derivative $f'(c)$:

$$f'(c) = \lim_{h\to 0} \frac{f(c+h) - f(c)}{h} = \lim_{x\to c} \frac{f(x) - f(c)}{x - c}.$$

The connection between the two limits is the relationship $x = c + h$, and $h \to 0$ is equivalent to $x \to c$.

Product Rule: Using that $f'(x)$, $g'(x)$ exist, we want to show

$$\mathrm{D}_x(f(x)g(x)) = f'(x)g(x) + f(x)g'(x)$$

In order to compute $\mathrm{D}_x(f(x)g(x))$ we need to compute

$$\lim_{h \to 0} \frac{f(x+h)g(x+h) - f(x)g(x)}{h}.$$

Notation: Often, if x is the input variable, we will call h, "delta x", or "change (deviation) in x", and write it Δx. There's a reason for this coming up. In any case, using the change in notation we need to compute

$$\lim_{\Delta x \to 0} \frac{f(x+\Delta x)g(x+\Delta x) - f(x)g(x)}{\Delta x}.$$

Continuing with the new notation, Call e.g. y the output variable for f and write $y = f(x)$ for the fixed input x and output $f(x)$ under consideration (as $\Delta x \to 0$); Call e.g. z, the output variable for g and write $z = g(x)$ for the fixed input x and output $g(x)$ under consideration (as $\Delta x \to 0$).

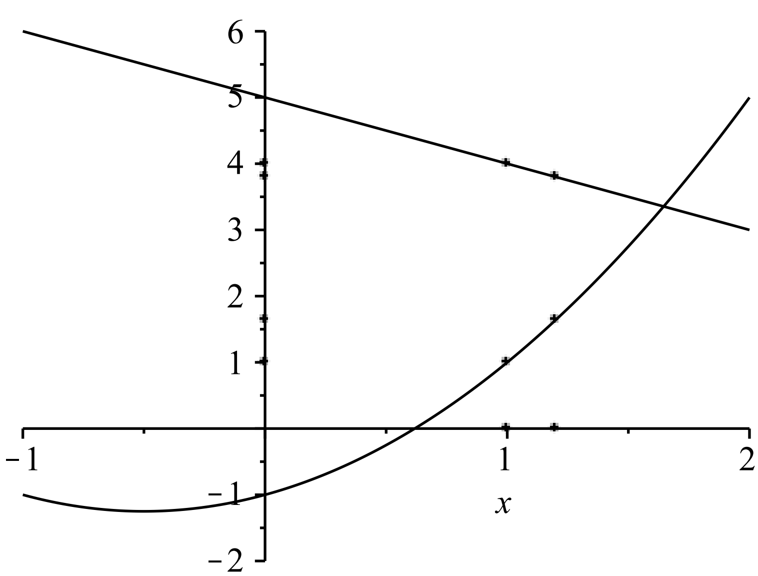
Then the change Δx in the input variable causes corresponding changes

$$\Delta y = f(x+\Delta x) - f(x) = f(x+\Delta x) - y$$
$$\Delta z = g(x+\Delta x) - g(x) = g(x+\Delta x) - z$$

in the output variables for f and g. In other words,

$$f(x+\Delta x) = y + \Delta y$$
$$g(x+\Delta x) = z + \Delta z\ .$$

Picture to fill in, to illustrate this new notation:



Finally, put all this together to get the product rule:

$$
\begin{aligned}
D_x(f(x)g(x)) &= \lim_{\Delta x \to 0} \frac{f(x+\Delta x)g(x+\Delta x) - f(x)g(x)}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{(y+\Delta y)(z+\Delta z) - yz}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{yz + (\Delta y)z + y(\Delta z) + (\Delta y)(\Delta z) - yz}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{(\Delta y)z + y(\Delta z) + (\Delta y)(\Delta z)}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} z + \lim_{\Delta x \to 0} y \frac{\Delta z}{\Delta x} + \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} \Delta z \\
&= f'(x)g(x) + f(x)g'(x) + f'(x) \cdot 0
\end{aligned}
$$

$\square$.

(In the last line the third term was zero because g differentiable implies g continuous, implies $\lim_{\Delta x \to 0} \Delta z = 0.$)

<u>Exercise 3)</u> Work out the quotient rule the same way we worked out the product rule!!! Here's the start ...

$$
\begin{aligned}
D_x\left(\frac{f(x)}{g(x)}\right) &= \lim_{\Delta x \to 0} \frac{\dfrac{f(x+\Delta x)}{g(x+\Delta x)} - \dfrac{f(x)}{g(x)}}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{\dfrac{y+\Delta y}{z+\Delta z} - \dfrac{y}{z}}{\Delta x} \quad \ldots
\end{aligned}
$$

(After we finish this we'll do more examples).