

Math 1210-001
Wednesday Apr 6
WEB L112

- Discuss the accumulation function, and why Part 1 of FTC is true (Tuesday notes). There are also two good examples at the end of those notes. Then
- Review day for exam 3. We can discuss any WebWork or lab problems from this week that you'd like hints for, or that confuse you. There are also worked examples in these notes, from earlier sections.
- Exam is this Friday April 8, 9:35 AM-10:35 AM WEB L112.
(We start 5 minutes early and end 5 minutes late, compared to usual meeting pattern.)
- No notes or collaboration allowed. You may use a scientific calculator (the sort that costs around \$15, e.g. a TI 30X). Symbolic answers will be accepted, so a calculator is not needed.
- The exam will be organized into 5 or so problems, with most problems having several parts.
- Your lab sessions tomorrow (Thursday) will be organized like the usual labs, except you'll be working on review problems - you'll be able to focus on the areas that concern you the most, in the lab. I'll post solutions to those review problems and to the lab you're handing in, by noon on Thursday.

The material you're responsible for on exam 3 is Chapter sections 3.4 - 4.4. The material for this exam implicitly requires what you've learned earlier in the course as well. Stars next to a topic means that there will be at least one problem that tests that section material directly. (A single problem can test more than one topic at a time.) Some of the topics without stars may also show up on the exam, so review these too.

3.4 Practical optimization problems *

3.5 Advanced graphing with Calculus - using asymptotes, limits and one-sided limits, first and second derivative information.

3.6 Mean Value Theorem

3.7 Solving equations with bisection method and/or Newton's method

3.8 Antiderivatives *

3.9 Differential equations *

4.1 Introduction to area

4.2 Riemann sums and definite integrals*

4.3 Properties of definite integrals, and part I of the Fundamental Theorem of Calculus *

4.4 Part II of the FTC *

3.4 Practical optimization problems

- Recognize the function to be optimized, and express it in terms of one or more variables. Sketch and label a picture if that will help.
- Use constraint equation(s) to rewrite the objective function as a function of a single variable.
- Use calculus to find the optimal solution.

Example: Two positive numbers add up to 8. What is the largest their product can be?

Solution: Call the numbers x, y . Then we wish to maximize the product xy .

objective function: $p = xy$.

constraint: $x + y = 8$.

From the constraint, $y = 8 - x$ so we can substitute for y and maximize

$$p(x) = x(8 - x) = 8x - x^2.$$

The natural domain for x is $0 \leq x \leq 8$. Since $p(0) = p(8) = 0$ the maximum must occur at a stationary point.

$$p'(x) = 8 - 2x.$$

$$p'(x) = 0 \Rightarrow x = 4.$$

So the maximum product is when $x = 4 \Rightarrow y = 4 \Rightarrow p = 16$. (You can also use first or second derivative information to justify that this is the maximum product.)

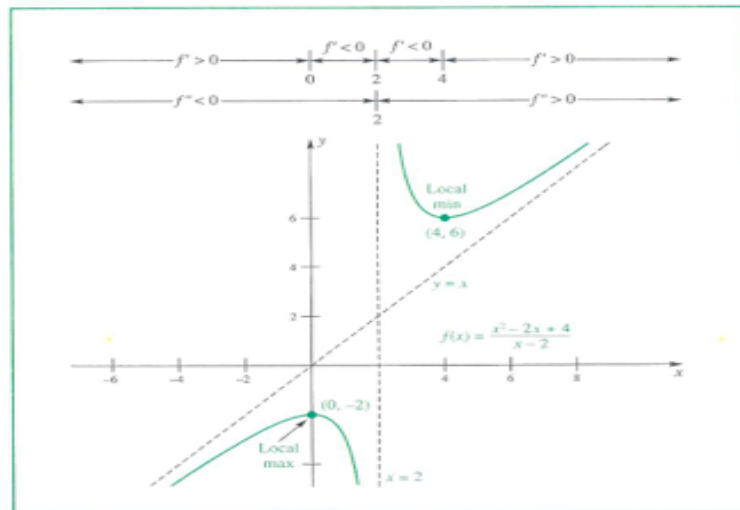
That example is probably too easy for an exam. There are lots of further worked examples available online, e.g. at

<https://www.math.ucdavis.edu/~kouba/CalcOneDIRECTORY/maxmindirectory/MaxMin.html>

3.5 Advanced graphing with Calculus.

- use asymptote information from limits and one-sided limits, and first and second derivative information, to graph functions.

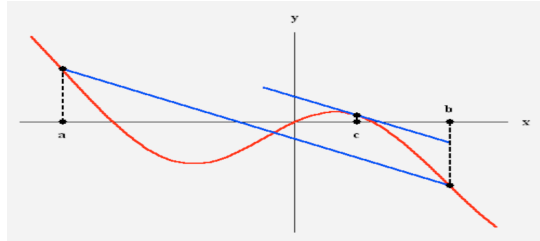
Example: $f(x) = \frac{x^2 - 2x + 4}{x - 2} = x + \frac{4}{x - 2}$, $f'(x) = \frac{x(x - 4)}{(x - 2)^2}$, $f''(x) = \frac{8}{(x - 2)^3}$.



3.6 Mean Value Theorem

- if f is continuous on $[a, b]$ and differentiable on (a, b) then there exists at least one c with $a < c < b$ so that

$$\frac{f(b) - f(a)}{b - a} = f'(c)$$



- The MVT explains why the only functions with identically zero derivative function are constants; so why once you know a single antiderivative of a function all other antiderivatives differ by a constant. MVT reappears in Calc 2.

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3.7 Solving equations numerically: bisection method (slow, but works for continuous functions); Newton's method (fast, but requires differentiable functions and a good first guess).

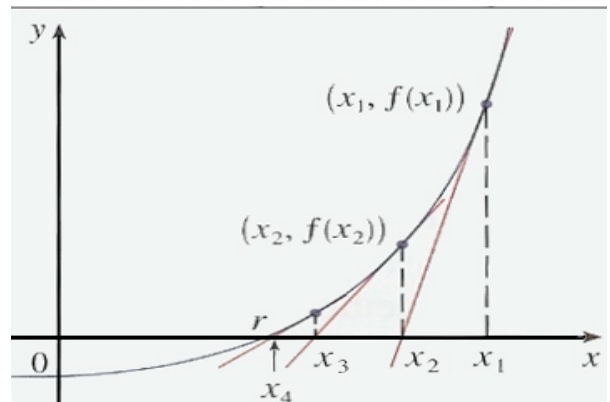
To find a root x solving $f(x) = 0$:

- bisection method: begin with interval $[a_1, b_1]$ for which $f(a_1)f(b_1) < 0$ (i.e. f has opposite signs at endpoints, so a root in between). Iterate as follows: given subinterval $[a_n, b_n]$ with $f(a_n)f(b_n) < 0$ let

$m_n = \left(\frac{a_n + b_n}{2} \right)$ and let $[a_{n+1}, b_{n+1}]$ be the choice of $[a_n, m_n]$ or $[m_n, b_n]$ for which $f(a_{n+1})f(b_{n+1}) < 0$.

- Newton's method. Let x_1 be a good initial guess. Let x_{n+1} be the x -intercept of the tangent line to the graph of f at $(x_n, f(x_n))$,

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} := g(x_n).$$



3.8 Antiderivatives

$$\int u^n du = \frac{u^{n+1}}{n+1} + C \quad (n \neq -1) \quad \text{power rule in reverse}$$

$$\int \cos(u) du = \sin(u) + C$$

$$\int \sin(u) du = -\cos(u) + C$$

$$\int f'(x) dx = f(x) + C$$

$$D_x \left(\int f(x) dx \right) = f(x)$$

$$\int kf(x) dx = k \int f(x) dx \quad \text{constant multiple rule in reverse}$$

$$\int f(x) + g(x) dx = \int f(x) dx + \int g(x) dx \quad \text{sum rule in reverse}$$

$$\int f(g(x))g'(x) dx = F(g(x)) + C \quad \text{chain rule in reverse } (F'(u) = f(u))$$

$$\int g(x)^n g'(x) dx = \frac{1}{n+1} g(x)^{n+1} + C \quad \text{generalized power rule, special case of previous rule}$$

- u -substitution for doing chain rule in reverse with less difficulty: for a constant multiple of $\int f(g(x))g'(x) dx$

$$\text{let } u = g(x), du = g'(x)dx$$

$$\int f(g(x))g'(x) dx = \int f(u) du = F(u) + C = F(g(x)) + C$$

3.9 Differential equations

- solving

$$\frac{dy}{dx} = f(x)$$

for $y(x)$ is just antidifferentiation:

$$y = \int f(x) dx$$

and the constant of integration can be adjusted to satisfy an initial condition $y(x_0) = y_0$.

- solving

$$y''(x) = f(x)$$

for $y(x)$ involves successive antidifferentiation to get $y'(x)$ and then $y(x)$. Part of the solution will consist of $C_1x + C_2$ from the successive constants of integration. Applications: reconstructing velocity and position functions from acceleration function (of time).

- solving the separable DE

$$\frac{dy}{dx} = \frac{f(x)}{g(y)}$$

by using the chain rule in reverse is accomplished with differential magic:

$$\begin{aligned} g(y) dy &= f(x) dx \\ \int g(y) dy &= \int f(x) dx \\ G(y) &= F(x) + C \end{aligned}$$

is the implicit solution, which may possibly be solved for $y(x)$ explicitly.

4.1 Area and summation notation

- algebra:

$$a_1 + a_2 + \dots + a_n = \sum_{i=1}^n a_i$$

$$\sum_{i=1}^n 1 = 1 + 1 + \dots + 1 = n$$

$$\sum_{i=1}^n c a_i = c \sum_{i=1}^n a_i$$

$$\sum_{i=1}^n (a_i + b_i) = \sum_{i=1}^n a_i + \sum_{i=1}^n b_i$$

"magic" formulas (will be provided as needed):

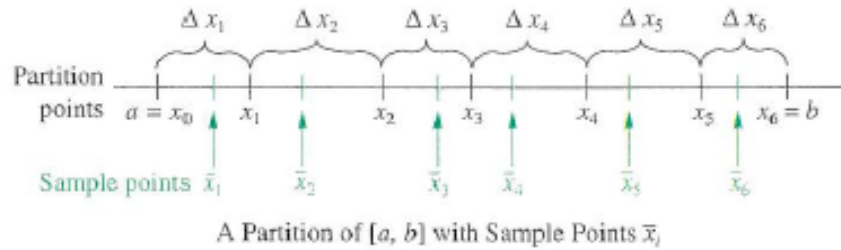
$$\sum_{i=1}^n i = 1 + 2 + 3 + \dots + n = \frac{n(n+1)}{2} = \frac{n^2}{2} + \frac{n}{2}$$

$$\sum_{i=1}^n i^2 = 1 + 2^2 + 3^2 + \dots + n^2 = \frac{n(n+1)(2n+1)}{6} = \frac{n^3}{3} + \frac{n^2}{2} + \frac{n}{6}$$

$$\sum_{i=1}^n i^3 = 1^3 + 2^3 + 3^3 + \dots + n^3 = \left[\frac{n(n+1)}{2} \right]^2$$

$$\sum_{i=1}^n i^4 = 1^4 + 2^4 + 3^4 + \dots + n^4 = \frac{n(n+1)(2n+1)(3n^2+3n-1)}{30}$$

4.2 Riemann integral



$$R_P = \sum_{i=1}^n f(\bar{x}_i) \Delta x_i.$$

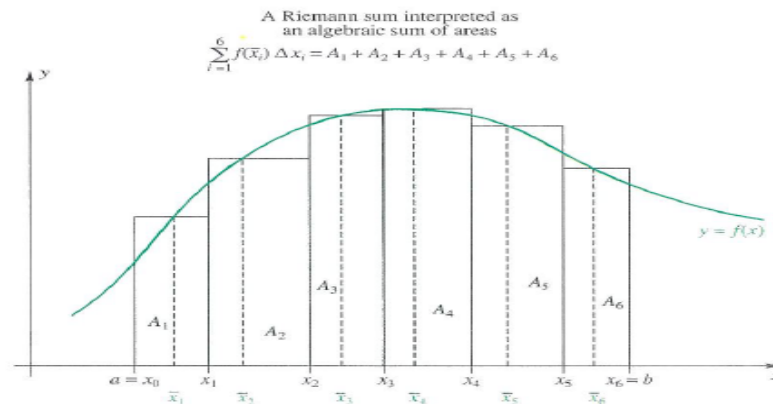


Figure 3

$$\int_a^b f(x) \, dx := \lim_{\max \Delta x_i \rightarrow 0} \sum_{i=1}^n f(\bar{x}_i) \Delta x_i$$

special case of n equal subintervals, using right endpoints:

$$\int_a^b f(x) \, dx = \lim_{n \rightarrow \infty} \sum_{i=1}^n f(x_i) \Delta x.$$

(With left endpoints the $f(x_i)$ would be replaced with $f(x_{i-1})$). The general template with n equal subdivisions expanded:

$$\Delta x = \frac{(b - a)}{n}$$

$$x_0 = a$$

$$x_1 = a + \Delta x = a + \frac{(b - a)}{n}$$

$$x_2 = a + 2 \Delta x = a + \frac{2(b - a)}{n}$$

$$x_i = a + i \Delta x = a + i \frac{(b - a)}{n}.$$

The i^{th} interval is

$$I_i = [x_{i-1}, x_i] = [a + (i-1)\Delta x, a + i(\Delta x)].$$

linearity properties:

$$\int_a^b c f(x) dx = c \int_a^b f(x) dx$$

$$\int_a^b f(x) + g(x) dx = \int_a^b f(x) dx + \int_a^b g(x) dx.$$

4.3 Accumulation functions and Part 1 of the FTC

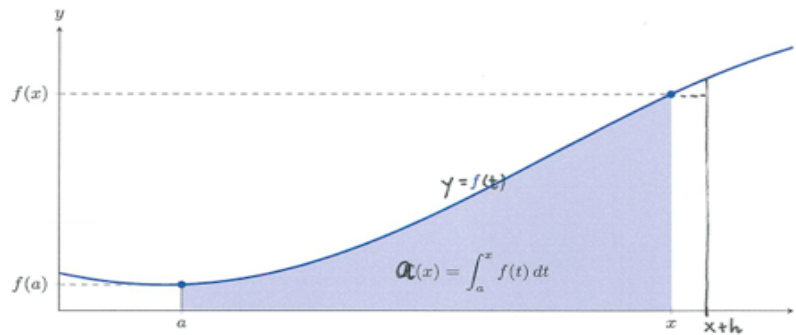
Let $f(x)$ be continuous on $[a, b]$. Define the definite integral on $[a, b]$ and subintervals using limits of Riemann sums. Define the "accumulation function" $\mathcal{A}(x)$ by integrating f from a to x :

$$\mathcal{A}(x) = \int_a^x f(t) dt$$

Part I of Fundamental Theorem of Calculus $\mathcal{A}(x)$ is an antiderivative of $f(x)$, i.e.

$$D_x \left(\int_a^x f(t) dt \right) = f(x).$$

(So all antiderivatives are of the form $\mathcal{A}(x) + C$.)



4.4 Part 2 of the FTC

Part II: Let $f(x)$ be continuous on $[a, b]$. Let $F(x)$ be any antiderivative of f on $[a, b]$. Then

$$\int_a^b f(x) dx = F(b) - F(a).$$

proof: By definition of the accumulation function,

$$\int_a^b f(x) dx = \mathcal{A}(b) = \mathcal{A}(b) - \mathcal{A}(a).$$

Since the accumulation function is an antiderivative of f , every other antiderivative $F(x) = \mathcal{A}(x) + C$, so

$$F(b) - F(a) = \mathcal{A}(b) - \mathcal{A}(a) = \int_a^b f(x) dx.$$

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