Math 1210-001
Tuesday Apr 5
WEB L110

4.2-4.4 continued

Today: We'll discuss properties of the definite integral (from the last pages of Monday's notes); discuss why Part 1 of the FTC is true; and work more examples, mostly at your discretion.

- Which WebWork or lab problems would you like to discuss at the end of class, time permitting?

- Discuss the properties of definite integrals on the last 2 pages of Monday's notes.

Exercise 1) To test the integral properties discussed in Monday's notes: Suppose

$$\int_1^3 f(x)\, dx = 6, \qquad \int_1^2 f(x)\, dx = 2.$$

Find

1a) $\int_2^3 f(x)\, dx.$

1b) $\int_1^3 4 f(x) + 2 x^2\, dx.$

Reversing limits of integration: If $a < b$ we define

$$\int_b^a f(x)\, dx = -\int_a^b f(x)\, dx.$$

Also

$$\int_a^a f(x)\, dx := 0.$$

These definitions are consistent with the FTC, since then

$$\int_c^d f(x)\, dx = F(d) - F(c)$$

regardless of whether $c < d$, $c > d$, $c = d$.

1c) Continuing with the first exercise, what is

$$\int_3^1 f(x)\, dx \ ?$$

- We'll discuss Part I of the FTC below (which also leads to another proof of Part II):

The Fundamental Theorem of Calculus: Let $f(x)$ be continuous on $[a, b]$. Define the definite integral on $[a, b]$ and subintervals using limits of Riemann sums. Define the "accumulation function" $\mathcal{A}(x)$ by integrating $f$ from $a$ to $x$:

$$\mathcal{A}(x) = \int_a^x f(t)\ \mathrm{d}t$$

Part I: $\mathcal{A}(x)$ is an antiderivative of $f(x)$, i.e.

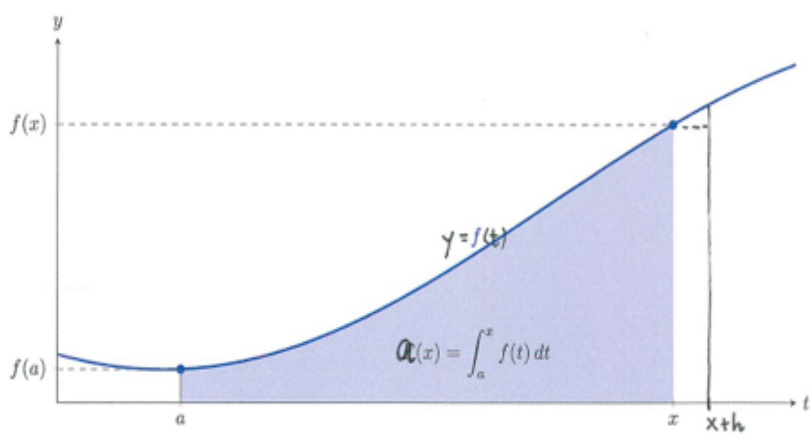
$$\mathrm{D}_x\left(\int_a^x f(t)\ \mathrm{d}t\right) = f(x).$$

(So all antiderivatives are of the form $\mathcal{A}(x) + C$.)

Part II: Let $F(x)$ be any antiderivative of $f$ on $[a, b]$. Then

$$\int_a^b f(x)\ \mathrm{d}x = F(b) - F(a).$$

For Part 1, here's a picture of the accumulation function, interpreted as an area function depending on the right endpoint. (As we saw yesterday, if the units of $f(t)$ were $\frac{\textit{distance}}{\textit{time}}$ and for $t$ were *time*, then the Riemann sums and accumulation function would be measuring accumulation of *distance*. Similarly, if $f(t)$ was measuring water flow in $\frac{\textit{volume}}{\textit{time}}$ then the accumulation function would be measuring accumulated *volume*.




Our goal for Part 1 is to show $\mathcal{A}'(x) = f(x)$. Let's use the limit definition of derivative:

$$\mathcal{A}'(x) = \lim_{h \to 0} \frac{\mathcal{A}(x+h) - \mathcal{A}(x)}{h}.$$

To make using the picture easier we'll assume $h > 0$ and show

$$\lim_{h \to 0+} \frac{\mathcal{A}(x+h) - \mathcal{A}(x)}{h} = f(x).$$

(One can show the equality for the left hand limit in a similar fashion.)
Notice that

$$\mathcal{A}(x+h) - \mathcal{A}(x) = \int_a^{x+h} f(t)\,dt - \int_a^x f(t)\,dt = \int_x^{x+h} f(t)\,dt$$

The idea of Part 1 is that $\int_x^{x+h} f(t)\,dt$ is very close to $h \cdot f(x)$ when $h$ is small, since (i) all of the $f(t)$ values on that short interval will be very close to the value $f(x)$ at the left endpoint; and (ii) the interval is $h$ units wide. Dividing by $h$ and taking the limit will give he desired derivative statement.

Here are the details: Let $m$ be the minimum value of $f(t)$ on the interval $[x, x+h]$ and let $M$ be the maximum value. (As $h \to 0$ both $m$ and $M$ will approach $f(x)$ since $f$ is continuous.) So by integral comparisons at the end of Monday's notes:

$$m\,h = \int_x^{x+h} m\,dx \le \int_x^{x+h} f(x)\,dx \le \int_x^{x+h} M\,dx = M\,h.$$

$$\Rightarrow m \le \frac{\mathcal{A}(x+h) - \mathcal{A}(x)}{h} \le M.$$

Now apply the squeeze Theorem: Since both $m$ and $M$ approach $f(x)$ as $h \to 0+$, we deduce

$$\lim_{h \to 0+} \frac{\mathcal{A}(x+h) - \mathcal{A}(x)}{h} = f(x).$$

□

That completes the proof of Part 1 of the FTC. It turns out that Part 2 follows very quickly from Part 1 (as an alternate to how we already understood Part 2):

Part II: Let $F(x)$ be any antiderivative of $f$ on $[a, b]$. Then

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

Proof: By definition

$$\int_a^b f(x)\,dx = \mathcal{A}(b) = \mathcal{A}(b) - \mathcal{A}(a)$$

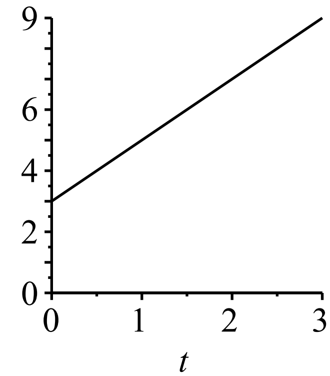
(since $\mathcal{A}(a) = 0$). But since $\mathcal{A}(x)$ is an antiderivative of $f(x)$ any other antiderivative $F(x) = \mathcal{A}(x) + C$ so

$$F(b) - F(a) = \mathcal{A}(b) - \mathcal{A}(a).$$

□

Exercise 2) Verify FTC Part 1 using geometry to compute $\mathcal{A}(x)$ and its derivative in the following example:

$$D_x \int_0^x 2\,t + 3 \, dt$$




Shortcut in using substitution for definite integrals:
We seek to evaluate

$$\int_a^b f(g(x))g'(x)\, dx$$

Method 1 (not the shortcut) use $u$-substitution, $u = g(x)$, $du = g'(x)dx$, so the indefinite integral

$$\int f(g(x))\, dx = \int f(u)\, du = F(u) + C = F(g(x)) + C$$

so

$$\int_a^b f(g(x))g'(x)\, dx = F(g(b)) - F(g(a)).$$

Method 2 (shortcut): change the limits (interval endpoints) to $u - limits$ at the same time you make the $u$-substitution. It yields the same correct answer as above.

$$\int_a^b f(g(x))g'(x)\, dx$$

Use the $u$-substitution $u = g(x)$, $du = g'(x)dx$. Also, when $x = a$, $u = g(a)$; when $x = b$, $u = g(b)$. Substitute these endpoints too:

$$\int_a^b f(g(x))g'(x)\, dx = \int_{g(a)}^{g(b)} f(u)\, du = F(g(b)) - F(g(a)).$$

Exercise 3) Compute

$$\int_0^3 x(x^2 + 1)^9\, dx.$$