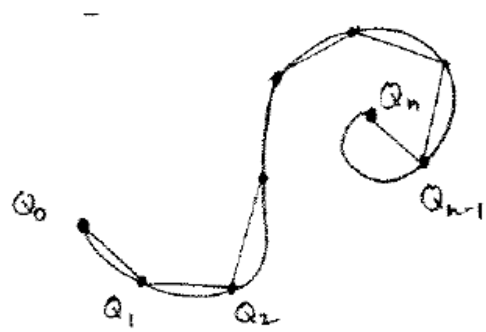
Math 1210-001
Wednesday Apr 20
WEB L112

- Friday quiz will be a volume of revolution question.
- Webwork or lab question clarifications?

Section 5.4: Lengths of curves and surface areas of revolution. (We'll cover curve lengths today.)

Mathematically (in the absence of string!) the length of a curve is a limit of approximate lengths:



$$\text{Approximate length} = \sum_{i=1}^{n} \textit{length of segment}\left(Q_{i-1}Q_i\right)$$

$$\text{Length} = \lim_{\max \textit{segment length} \to 0} \left(\sum_{i=1}^{n} \textit{length of segment}\left(Q_{i-1}Q_i\right) \right)$$

provided the limit exists. As we'll see, this limit definition leads to definite integrals since the sums above will end up resembling Riemann sums.

But, what exactly do we mean by a "curve"?

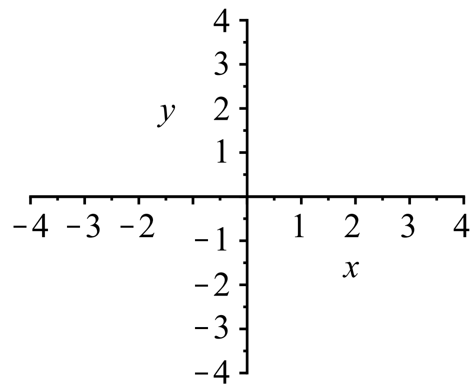
Definition A parametric curve in the plane is the collection of points (x, y) with

$$x = f(t)$$
$$y = g(t)$$
$$a \le t \le b$$

where *f*, *g* are continuous functions. In this case "t" is called the parameter. Often we think of "t" as time and think of the points $(f(t), g(t))$ as the coordinates of the position at time *t*, of a particle moving along the curve.

Exercise 1 Describe geometrically and sketch the curve given parametrically by

$$x = 3\cdot\cos(t)$$
$$y = 3\cdot\sin(t)$$
$$0 \le t \le \frac{3}{2}\pi$$



Some special parametric curves are just graphs. For example

$$x = t$$
$$y = f(t)$$
$$a \le t \le b$$

just describes the graph $y = f(x)$, where $x = t$ is the parameter.

And

$$x = g(t)$$
$$y = t$$
$$c \le t \le d$$

describes the graph $x = g(y)$, where $y = t$ is the parameter.

As Exercise 1 indicates, many parametric curves are not graphs of one variable or the other.

Approximation of curve length from Exercise 1, with a 100-point subdivision. Subdivide the inverval

$$0 \le t \le \frac{3}{2}\pi$$

into 100 subintervals of length

$$\Delta t = \frac{1.5 \cdot \pi}{100}$$

with partition points

$$t_i = 0 + i \cdot \Delta t, \quad i = 0, 1, 2, \ldots. n$$

and corresponding curve points

$$x_i = 3 \cdot \cos\left(t_i\right), y_i = 3 \cdot \sin\left(t_i\right).$$

Here's the approximate length computation, the decimal value of exact length (from geometry), and a corresponding picture of the points on the curve:

> $\Delta t := evalf\left(\frac{1.5 \cdot \pi}{100}\right):$
> **for** i **from** 0 **to** 100 **do**
> $x[i] := 3 \cdot \cos(i \cdot \Delta t):$
> $y[i] := 3 \cdot \sin(i \cdot \Delta t):$
> **end do**:

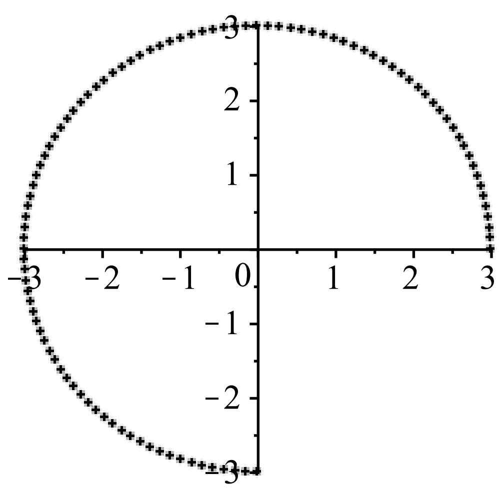
> $\sum_{j=1}^{100} \sqrt{(x[j] - x[j-1])^2 + (y[j] - y[j-1])^2};$

$$14.13585867 \qquad \textbf{(1)}$$

> $evalf\left(\frac{3}{4} \cdot 2 \cdot \pi \cdot 3\right);$ *#"exact" decimal*

$$14.13716694 \qquad \textbf{(2)}$$

> *with(plots)* :
> $plot([3 \cdot \cos(t), 3 \cdot \sin(t), t = 0..1.5 \cdot \pi], style = point, numpoints = 100, color = black);$



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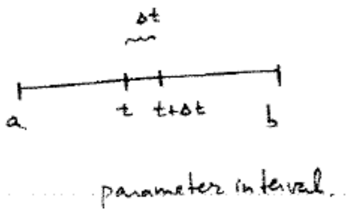
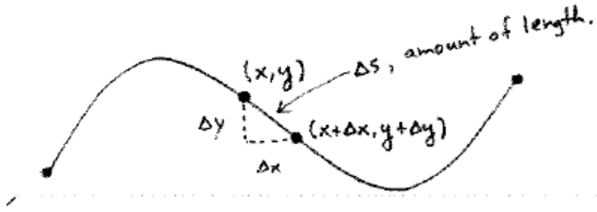
How to compute curve lengths with Calculus:

$x = f(t)$ $\quad$ $x + \Delta x = f(t + \Delta t)$

$y = g(t)$ $\quad$ $y + \Delta y = g(t + \Delta t)$





$$\Delta s \approx \text{hypotenuse} = \sqrt{(\Delta x)^2 + (\Delta y)^2} = \begin{cases} \sqrt{\left(\frac{\Delta x}{\Delta t}\right)^2 + \left(\frac{\Delta y}{\Delta t}\right)^2}\, \Delta t & x = f(t),\ y = g(t) \\ \sqrt{1 + \left(\frac{\Delta y}{\Delta x}\right)^2}\, \Delta x & x = x,\ y = g(x) \quad a \le x \le b \\ \sqrt{\left(\frac{\Delta x}{\Delta y}\right)^2 + 1}\, \Delta y & x = f(y),\ y = y \quad c \le y \le d \end{cases}$$

for a partition of $[a,b]$,

$$L \approx \sum_{i=1}^{n} \Delta s_i$$

$$L = \lim_{\|P\| \to 0} \sum_{i=1}^{n} \Delta s_i$$

$$\boxed{L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt},$$

with analogous formulas for graphs

For $y = f(x)$, $a \le x \le b$, i.e. $x = x$, $y = f(x)$ in the formula above:

$$L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, \mathrm{d}x$$

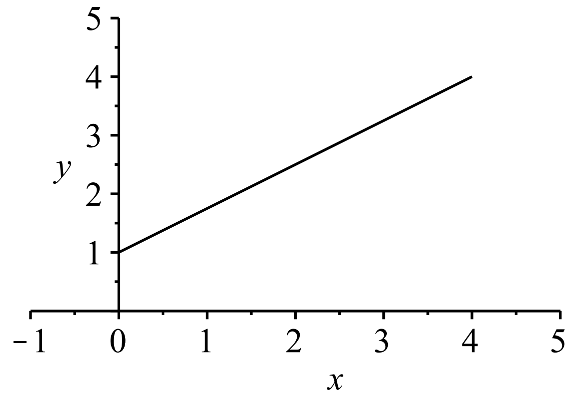
For $x = g(y)$, $c \le x \le d$, i.e. $x = g(y)$, $y = y$ in the formula above:

$$L = \int_c^d \sqrt{\left(\frac{dx}{dy}\right)^2 + 1}\, \mathrm{d}y$$

Exercise 2) Use the curve length formula from the previous page to compute the length of the arc from Exercise 1:

$$x = 3\cdot\cos(t),\, y = 3\cdot\sin(t),\, 0 \le t \le \frac{3}{2}\cdot\pi.$$

Exercise 3) Consider the line segment from $(0, 1)$ to $(4, 4)$.



3a) Find the length using the Pythagorean Theorem.

3b) Express the line as the graph $y = f(x)$ and find the length with one of the integrals on the previous page.

3c) Express the line as a graph $x = g(y)$ and find the length using one of the integrals on the previous page.