1 Regular Perturbation Theory

(due Oct. 3, 2019)

1. Find a two-term asymptotic expansion, for small ϵ, for all solutions of the following equations:

   (a)
   $$\epsilon x^3 - 3x + 1 = 0 \tag{1}$$

   (b)
   $$x^{2+\epsilon} = \frac{1}{x + 2\epsilon} \tag{2}$$

   (c)
   $$xe^{-x} = \epsilon \tag{3}$$

   (d)
   $$x^3 - x^2 - \epsilon = 0 \tag{4}$$

2. Find a two-term asymptotic expansion, for small ϵ, of the solution of the following boundary value problems:

   (a)
   $$y'' + y + \epsilon y^3 = 0, \qquad y(0) = 0, \qquad y(\frac{\pi}{2}) = 1. \tag{5}$$

   (b)
   $$y'' - y = 0, \qquad y(0) = 0, \qquad y(1 + \epsilon) = 1 \tag{6}$$

   Hint: Use a change of variables to transform this to a problem on a fixed domain (independent of ϵ).

3. Suppose that a one-dimensional cell that is 10 μm long and the voltage potential is held fixed at both ends (so that $\Phi = 0$ at the ends). Suppose that the cell contains approximately equal amounts (say 400mM) of sodium and chloride, but suppose there is a slight excess of sodium. How much excess sodium is there if the total difference between maximal and minimal potential in the cell is 0.1mV? Plot the (approximate) distribution of potential and ion concentrations for both species.

   Hint: Find an approximate solution of the Poisson-Boltzman equation which is derived as follows: According to the Poisson equation,

   $$\epsilon \nabla^2 \phi = -qN_a \sum_i z_i c_i \tag{7}$$

where ϵ is the dielectric constant of the medium, N_a is Avagadro's number, q is the unit electric charge, c_i and z_i are the concentration and unit charge for the ith species. According to the Nernst-Planck equation,

$$0 = \nabla c_i + \frac{z_i F}{RT} c_i \nabla \phi. \tag{8}$$

Use this information to derive the Poisson-Boltzmann equation in dimensionless units

$$\nabla^2 \Phi = -\beta \sum_i z_i \alpha_i \exp(-z_i \Phi), \tag{9}$$

where

$$\beta = q \frac{L^2 F N_A}{\epsilon RT} C_0, \tag{10}$$

and C_0 is a characteristic concentration, L is the length of a one dimensional domain. What is α_i? For parameter values, use that $\epsilon = 6.938 \times 10^{-10} \text{C}^2/\text{Nm}^2$ at 25C, $q = 1.6 \times 10^{-19}\text{C(oulombs)}$, $\frac{RT}{F} = 25.8\text{mV}$. How large is β for this cell?

Use that the net charge

$$\delta = \sum_i z_i \alpha_i. \tag{11}$$

is a small dimensionless parameter to find the asymptotic solution of the problem.

4. Approximate the time it takes for a projectile whose terminal velocity is V_T with height $y(t)$ above the earth, with $y(0) = 0$ and $y_t(0) = V_0$, to return to earth, assuming that $\frac{V_0^2}{gR}$ and $\frac{V_0}{V_T}$ are small. What is the effect of drag on the return time?

Hint: Use the equation

$$y_{tt} = -\frac{g}{(1 + \frac{y}{R})^2} - \frac{gR}{V_T(R + y)} y_t. \tag{12}$$

to account for viscous atmospheric drag.

2 Matched Asymptotic Expansions

(due Oct. 3, 2019)

1. Find a composite expansion for the solutions of each of the following (include a plot of the approximate solution):

   (a)
   $$\epsilon y'' + y' = a(> 0), \qquad y(0) = 0, \qquad y(1) = 1, \tag{13}$$

   (b)
   $$\epsilon y'' + 2y' + y^3 = 0, \qquad y(0) = 0, \qquad y(1) = \frac{1}{2} \tag{14}$$

(c)
$$\epsilon y'' - y' - y^2 = 1, \qquad y(0) = \frac{1}{3}, \qquad y(1) = 1 \tag{15}$$

(d)
$$\epsilon y'' = yy' - y^3, \qquad y(0) = \frac{3}{5}, \qquad y(1) = -\frac{2}{3} \tag{16}$$

(e)
$$\epsilon y'' = 9 - (y')^2, \qquad y(0) = 0, \qquad y(1) = 1 \tag{17}$$

3 QSS Analysis

(due Nov.15, 2019)

1. The calcium-activated potassium channel has two open states and two closed states (see Fig. 1). The channel has two binding sites for calcium and can open when one of the states is occupied and remains open when the second site is bound. The binding process is a fast process, while the transition between open and closed states is slow. Find the equation describing slow dynamics of the opening of the channel as a function of calcium concentration.
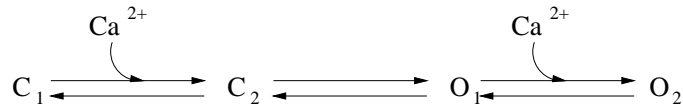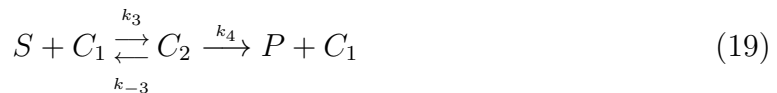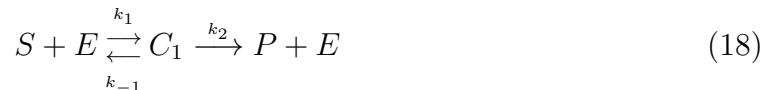

Figure 1: Diagram of the four states for the calcium-activated potassium channel.

2. Suppose that an enzyme can bind two substrate molecules, so can exist in one of three states, a free moleule S, a complex with one molecule bound C_1, and a complex with two molecules bound C_2. The corresponding chemical reactions are

$$S + E \underset{k_{-1}}{\overset{k_1}{\rightleftharpoons}} C_1 \xrightarrow{k_2} P + E \tag{18}$$

$$S + C_1 \underset{k_{-3}}{\overset{k_3}{\rightleftharpoons}} C_2 \xrightarrow{k_4} P + C_1 \tag{19}$$

Give a "proper" derivation of the corresponding Michaelis-Menten rate of production of product P, using the assumption that the amount of substrate is much larger than the amount of complex or enzyme.

3. Consider the process of autocatalytic conversion of a substrate to an enzyme given by

$$S + E \underset{k_{-1}}{\overset{k_1}{\rightleftharpoons}} C \xrightarrow{k_2} E + E \tag{20}$$

Find the (slow) rate of substrate conversion under the assumption that $k_2 \ll k_{-1}$.

4. Suppose a particle randomly switches between three states

$$S_{-1} \overset{k}{\underset{k}{\rightleftarrows}} S_0 \overset{k}{\underset{k}{\rightleftarrows}} S_1, \tag{21}$$

and in states S_j it moves with velocity jV, $j = -1, 1$, while in state S_0 the particle diffuses with diffusion coefficient D. Suppose the switching rates k are quite large. Find the effective diffusion coefficient for this particle.

4 Multiscale/Averaging and Homogenization

(due Dec. 12, 2019)

1. Using your choice of method (multiscale analysis or averaging), find approximate solutions for the equations

    (a)
    $$u'' + \epsilon(u')^3 + u = 0 \tag{22}$$

    (b)
    $$u'' - \epsilon \cos t (u')^2 + u = 0 \tag{23}$$

    (c)
    $$u'' + \epsilon \cos(\epsilon t)(u')^3 + u = 0 \tag{24}$$

    (d)
    $$u'' + \epsilon(\epsilon + u')u' + u = 0 \tag{25}$$

    (e)
    $$u'' + \epsilon(\epsilon + (u')^2)u' + u = 0 \tag{26}$$

    Remark: For this problem, use higher order averaging. Why is this method preferable for this problem?

2. A laminate material consists of many thin layers of alternating materials with different thermal conductivities. Suppose the layers have thermal conductivities D_i and thicknesses l_i, $i = 1, 2$. Find the effective conductivity tensor for the composite material.

3. Find the effective diffusion tensor for a periodic medium in which the diffusion coefficient is rapidly varying in space, according to

$$D(x, y) = \exp\left(\alpha(\frac{x}{\epsilon}) + \beta(\frac{y}{\epsilon})\right), \tag{27}$$

    where $\alpha(x)$ and $\beta(y)$ are periodic functions with period a and b, respectively.

4. Find the coupled phase equations for a system of coupled lambda-omega $(\lambda - \omega)$ systems, of the form

$$\frac{d}{dt}\begin{pmatrix} x_i \\ y_i \end{pmatrix} = \begin{pmatrix} \lambda(r_i) & -\omega \\ \omega & \lambda(r_i) \end{pmatrix} \begin{pmatrix} x_i \\ y_i \end{pmatrix} + \epsilon \begin{pmatrix} \sum_j g_{ij} \\ 0 \end{pmatrix}, \tag{28}$$

    $i = 1, \cdots, N$, where $r_i = x_i^2 + y_i^2$, $\epsilon \ll 1$, $\lambda(1) = 0$, $\lambda'(1) < 0$, and

(a)

$$g_{ij} = c_{ij}(x_j - x_i), \tag{29}$$

(diffusive coupling)

(b)

$$g_{ij} = c_{ij}x_j \tag{30}$$

(synaptic coupling).