Name: _____

1. Consider the following linear drag, initial value problem, which could arise in studying the velocity of a certain object falling through air, dropped from a high altitude at time $t = 0$. In this case we have chosen "down" to be the positive direction, so the acceleration of gravity is (positive) $10 \ \frac{m}{s^2}$:

$$v'(t) = 10 - 0.2v \qquad v(0) = 0.$$

(a) Draw the phase diagram for velocity. Find and identify the terminal velocity.

(b) Solve the initial value problem, and verify that your solution is consistent with the phase diagram as $t \to \infty$.

(c) What percentage of the terminal velocity is $v(5)$?

2. Consider the same IVP as in problem 1,

$$v'(t) = 10 - 0.2v \qquad v(0) = 0.$$

Apply the numerical methods listed below to approximate the solution function on the time interval $0 \leq t \leq 1$. Do these computations by hand (using a calculator or other technology to do each step, but not using programmed code). You should use programmed code to check your work before you hand it in. I suggest using the MATLAB file `numerical.m` which is posted on the course web site. Make sure to record your answers to each part to at least four decimal places to accurately compare the numerical methods to the exact solution.

 (a) Euler: use step size $h = 0.5$, i.e. 2 time steps.

 (b) Improved Euler: use step size $h = 0.5$, i.e. 2 time steps.

 (c) Runge–Kutta: use step size $h = 0.5$, i.e. 2 time steps.

 (d) Compare these approximations for $v(1)$ with the exact solution at $t = 1$ and comment about the relative accuracy of the three techniques, by computing the relative errors

$$\left| \frac{v_{\text{approx}} - v_{\text{exact}}}{v_{\text{exact}}} \right|$$

in each case. (The step sizes are so large that you don't expect any of the techniques to be extremely accurate.)

3. If we drop a package from a helicopter with a parachute attached, the wind resistance or drag provided by the parachute does not quite depend linearly on the velocity. Assume that for a certain type of parachute the initial value problem is modeled by a somewhat more complicated equation of motion than that of the first two problems:

$$v'(t) = 10 - 0.2v - 0.27v^{1.5} \qquad v(0) = 0.$$

(a) What is the terminal velocity in this model? Notice that the "slope function" $10 - 0.2v - 0.27v^{1.5}$ is a decreasing function of v for $v > 0$; and that its value is 10 when v=0, and that its limit as $v \to \infty$ is $-\infty$. Thus the slope function has exactly one root which you should find using either the Maple or MATLAB `solve` command.

(b) This is a differential equation which does *not* have an elementary solution. Use the supplied MATLAB script `numerical.m` to compute approximate solutions on the interval $0 \leq t \leq 5$ using Euler, Improved Euler, and Runge–Kutta methods, with time steps $h = 0.5$ and $h = 0.05$. Plot your results and comment about the apparent accuracy of the three methods with these different choices of time step size.

(c) Since we don't know the exact solution function for this problem, there is not a quick, direct way to measure how accurate our approximate solutions are. There are error estimates which some of you may learn in later numerical analysis classes (analogous to error estimates you may have learned for Taylor series), but for the purposes of this lab undertake the following crude test: Exhibit and compare the numerical approximations for velocity at $t = 2$ and $t = 5$, for Runge-Kutta with $h = 0.05$ and $h = 0.01$. If these approximations are close (and the approximate solution graphs appear to line up exactly), then it's likely that they are also close to the exact solution.

(d) Based on your work above, approximately what percent of the terminal velocity is attained at $t = 2$ and $t = 5$ seconds?