Name: ____________________

1. Consider a 3×3 matrix

$$\mathbf{A} = \begin{bmatrix} a & 1 & 1 \\ 1 & a & 1 \\ 1 & 1 & a \end{bmatrix}.$$

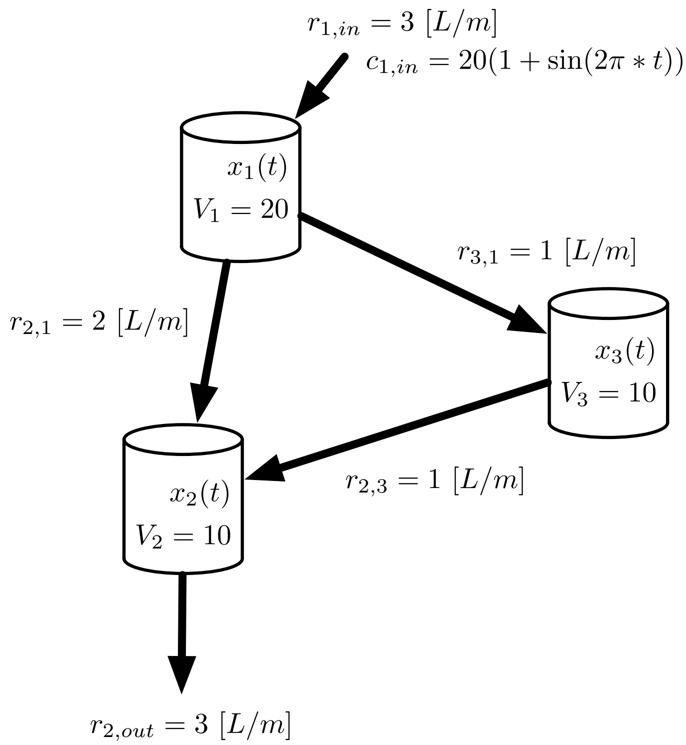
Set $a = 2$

(a) Find the eigenvalues and eigenvectors of the matrix $\mathbf{A}$. For each eigenvalue, find as many linearly independent eigenvectors as possible that correspond to it. (Hint: Part(a) may be useful in solving the characteristics equation for eigenvalues.)

(b) Find a basis for each eigenspace of $\mathbf{A}$.

(c) Given a square matrix $\mathbf{B}$ that is not invertible, explain why $\mathbf{B}$ must have a zero eigenvalue.

2. Consider the network of tanks shown in the figure below.



(a) Let

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, \quad \text{and} \quad \mathbf{c}_{in} = \begin{pmatrix} c_{1,in} \\ c_{2,in} \\ c_{3,in} \end{pmatrix}$$

denote the vector of concentrations in each tank and the vector concentrations going into each tank, respectively. Derive the system of differential equations of the form

$$\mathbf{x}'(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{c}(t)$$

(b) Find the eigenvalues and eigenvectors of $\mathbf{A}$.

(c) Write down the homogenous solution $\mathbf{x}_h$ of the system.

(d) Using the method of determined coefficients, one may guess the particular solution is of the form $\mathbf{d} + \mathbf{e}\sin 2\pi t + \mathbf{f}\cos 2\pi t$, where $\mathbf{d}$, $\mathbf{e}$, $\mathbf{f}$ are constant vectors. Find the vector $\mathbf{d}$ only. (No need to find for $\mathbf{e}$ and $\mathbf{f}$!)

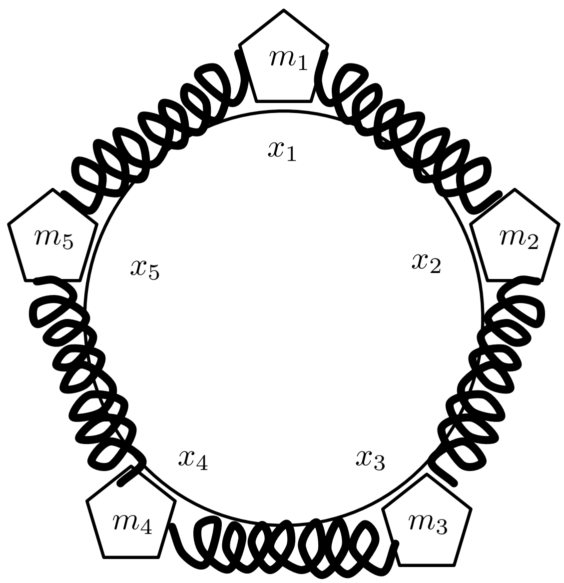
(e) Write the full general solution to the system in (a). (You may use the symbols $\mathbf{d}$, $\mathbf{e}$ and $\mathbf{f}$ in (d) in your answer.)

3. Consider the multiple mass-spring system shown in the figure below. We neglect the curvature so that each mass is pulled in opposite directions by the springs that connect its two neighbors. Recall that in an ordinary single mass-spring system that the restorative force is proportional to the change in the length of the spring so that

$$mx'' = -kx$$

Imagine, now that there are two masses, m_1 and m_2, connected by a single spring with spring constant k. We then have the system

$$m_1 x_1'' = -k(x_1 - x_2)$$
$$m_2 x_2'' = -k(x_2 - x_1)$$



(a) Suppose that all 5 springs have the same spring constant, k. Use the above information to construct a system of differential equations for the system of the form

$$\mathbf{M}\mathbf{x}'' = \mathbf{K}\mathbf{x}$$

where

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix}$$

Specify the matrices $\mathbf{M}$ and $\mathbf{K}$.

(b) Let $m_i = 1$ for $i = 1, ..., 5$. This reduces the above equation to $\mathbf{x}'' = \mathbf{K}\mathbf{x}$. Let $\mathbf{y} = \mathbf{x}'$. Convert the 5×5 system from part (a) to a 10×10 system for the vector

$$\mathbf{z} = \begin{pmatrix} \mathbf{x} \\ \mathbf{y} \end{pmatrix} = \begin{pmatrix} x_1 \\ \vdots \\ x_5 \\ y_1 \\ \vdots \\ y_5 \end{pmatrix}$$

This will have the form

$$\mathbf{z}' = \mathbf{A}\mathbf{z}$$

where $\mathbf{A}$ is a 10×10 matrix.

(c) Suppose we have an $2n \times 2n$ matrix

$$\mathbf{A} = \begin{pmatrix} \mathbf{B} & \mathbf{C} \\ \mathbf{D} & \mathbf{E} \end{pmatrix}$$

where $\mathbf{B}, \mathbf{C}, \mathbf{D}, \mathbf{E}$ are all 2×2 matrices. Then, if $\mathbf{DE} = \mathbf{ED}$, we have the result that

$$\det(\mathbf{A}) = \det(\mathbf{BE} - \mathbf{CD}).$$

Use this fact to show that if λ is an eigenvalue of the $\mathbf{K}$ in part (a), then $\omega = \sqrt{\lambda}$ is an eigenvalue of the matrix $\mathbf{A}$ from part (b).

(d) Find the eigenvalues of $\mathbf{K}$. What are all the possible frequencies that the system can oscillate at? Without actually computing the eigenvectors, write down the general solution of the system in terms of linear combination of functions of t times the eigenvectors. You may use any technology you want to find the eigenvalues.

(e) Find a non-trivial equilibrium solution to the system in (a).