Math 1220 Practice Final Exam April 27, 2012

Name: Solutions

Instructions:

- Answer the questions in the space provided.
- You must show your work in order to get credit! Writing just an answer is worth 0 points, even if the answer is correct.
- Partial credit will be awarded.
- The instructor has extra scratch paper if you need it.
- Graphing and scientific calculators are allowed, but smartphones and computers are not allowed.
- This exam is closed book and closed notes, except you may use two double sided 8.5 by 11 inch page of notes.

1. [10 points] Suppose $a, b > 0$, compute the following integral:

$$\int \frac{1}{ax - b}\, dx.$$

let $u = ax - b$

$\frac{1}{a} du = dx$

$$\int \frac{1}{ax-b} dx = \frac{1}{a} \int \frac{1}{u} du = \boxed{\frac{1}{a} \ln |ax-b| + C}$$

2. [10 points] Compute

$$\int \frac{6e^{1/x}}{x^2}\, dx.$$

let $u = \frac{1}{x} = x^{-1}$

$du = -x^{-2} dx = \frac{-1}{x^2} dx$

$$6 \int \frac{e^{1/x}}{x^2} dx = -6 \int e^u du = \boxed{-6e^{1/x} + C}$$

✓

3. [10 points] Completely solve the initial value problem (differential equation), showing all steps.

$$\frac{dy}{dx} = ky, \qquad y(0) = y_0$$

This differential equation is separable.

$$\frac{dy}{dx} = ky \implies \frac{dy}{y} = k\,dx \implies \int \frac{dy}{y} = \int k\,dx$$

$$\implies \ln|y| = kx + C$$

$$e^{\ln|y|} = e^{(kx+C)}$$ (Recall e^x and $\ln x$ are inverses.)

$$|y| = e^{kx} \cdot e^{C}$$ (Since $e^{kx} > 0$ for all $x \in \mathbb{R}$, we can drop the absolute value on y.)

$$y = e^{kx} \cdot e^{C}$$

$$\implies y(x) = e^{C} \cdot e^{kx}$$ (Now apply the initial condition, $y(0) = y_0$, to solve for e^C.)

$$y(0) = e^{C} \cdot e^{0} = y_0$$

$$\implies e^{C} \cdot 1 = y_0$$

Thus $\boxed{y(x) = y_0 e^{kx}}$ is the general solution

✓

4. [10 points] Use integration by parts to compute

$$\int \arctan(x)\,dx.$$

$u = \arctan(x)$ $\quad dv = dx$

$du = \frac{1}{1+x^2}\,dx$ $\quad v = x$

$u = 1+x^2$

$\frac{1}{2}\,du = x\,dx$

$$\int \arctan(x)\,dx = x\cdot\arctan(x) - \int \frac{x}{1+x^2}\,dx$$

$$= x\cdot\arctan(x) - \frac{1}{2}\int \frac{1}{u}\,du$$

$$= \boxed{x\cdot\arctan(x) - \frac{1}{2}\ln(1+x^2) + C}$$

5. [10 points] Use integration by parts to compute

$$\int \ln(x)\,dx.$$

$u = \ln x$ $\quad dv = dx$

$du = \frac{1}{x}\,dx$ $\quad V = X$

$$\int \ln x\,dx = x\cdot\ln x - \int x\cdot\frac{1}{x}\,dx$$

$$= x\cdot\ln x - \int dx$$

$$= \boxed{x\cdot\ln x - x + C}$$

✓

6. [10 points] Use the tabular method (i.e. integration by parts) to find:

$$\int \underbrace{x^3}_{u} \underbrace{e^{2x}\, dx}_{dv}.$$

du	$\int dv$	(+/-)
x^3	e^{2x}	$-$
$3x^2$	$\frac{1}{2}e^{2x}$	$+$
$6x$	$\frac{1}{4}e^{2x}$	$-$
6	$\frac{1}{8}e^{2x}$	$+$
0	$\frac{1}{16}e^{2x}$	$-$

$$\int x^3 e^{2x} dx = \frac{1}{2}x^3e^{2x} - \frac{3}{4}x^2e^{2x} + \frac{6}{8}xe^{2x} - \frac{6}{16}e^{2x} + C$$

$$= \boxed{e^{2x}\left(\frac{1}{2}x^3 - \frac{3}{4}x^2 + \frac{3}{4}x - \frac{3}{8}\right) + C}$$

✓

7. [10 points] Use a trigonometric substitution to integrate

$$\int \frac{dx}{x^2\sqrt{9-x^2}}.$$

Trig substitution:

$$\boxed{a^2 - x^2 \implies x = a\sin\theta}$$

$$3^2 - x^2 \implies x = 3\sin\theta \qquad dx = 3\cos\theta\, d\theta$$

$$\int \frac{dx}{x^2\sqrt{3^2-x^2}} = \int \frac{3\cos\theta\, d\theta}{3^2\sin^2\theta\sqrt{3^2 - 3^2\sin^2\theta}}$$

$$= \frac{1}{3}\int \frac{\cos\theta\, d\theta}{(\sin^2\theta)\cdot 3\sqrt{1-\sin^2\theta}}$$

Recall:

$$\sin^2\theta + \cos^2\theta = 1$$

$$1 - \sin^2\theta = \cos^2\theta$$

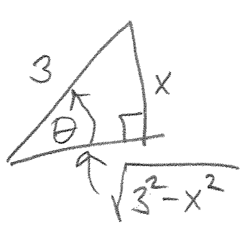
$$= \frac{1}{3^2}\int \frac{\cos\theta\, d\theta}{\sin^2\theta\sqrt{\cos^2\theta}}$$

$$= \frac{1}{3^2}\int \frac{\cancel{\cos\theta}\, d\theta}{\sin^2\theta\cdot\cancel{\cos\theta}}$$

$$= \frac{1}{3^2}\int \csc^2\theta\, d\theta$$

$$= \frac{1}{3^2}(-\cot\theta) + C$$

Now convert from $\theta \to x$ using $x = 3\sin\theta \implies \sin\theta = \frac{x}{3}$



$$\cot\theta = \frac{\text{adj.}}{\text{opp.}} = \frac{\sqrt{3^2-x^2}}{x}$$

$$\implies \boxed{\frac{-1}{9}\frac{\sqrt{9-x^2}}{x} + C}$$

✓

8. [10 points] Use l'Hôpital's rule to determine the following limit with indeterminate form: $\frac{0}{0}$.

$$\lim_{x \to 0} \frac{\tan x}{x}$$

$$\lim_{x\to 0} \frac{\tan x}{x} \overset{H}{=} \lim_{x\to 0} \frac{\sec^2 x}{1} = \lim_{x \to 0} \frac{1}{\cos^2 x} = \boxed{1}$$

9. [10 points] Use l'Hôpital's rule to determine the following limit with indeterminate form: $(\infty - \infty)$.

$$\lim_{x \to 0} \left(\csc x - \frac{1}{x} \right)$$

We must rewrite $\csc x - \frac{1}{x}$ into a form for which l'Hôpital's rule applies, i.e. either $\frac{0}{0}$ or $\frac{\infty}{\infty}$.

$$\csc x - \frac{1}{x} = \frac{1}{\sin x} - \frac{1}{x} = \frac{x}{x\sin x} - \frac{\sin x}{x \sin x} = \frac{x - \sin x}{x \sin x} \sim \frac{0}{0} \checkmark$$

$$\lim_{x\to 0}\left(\csc x - \frac{1}{x}\right) = \lim_{x \to 0} \frac{x - \sin x}{x \sin x} \overset{H}{=} \lim_{x\to 0} \frac{1-\cos x}{x\cos x + \sin x} \sim \frac{0}{0}$$

$$\overset{H}{=} \lim_{x \to 0} \frac{\sin x}{-x\sin x + \cos x + \cos x}$$

l'Hôpital's rule can be applied a 2nd time.

$$= \frac{0}{0+1+1}$$

$$= \frac{0}{2} = \boxed{0}$$

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10. [10 points] Evaluate the improper definite integral:

$$\int_{-\infty}^{\infty} xe^{-x^2}\,dx.$$

This is 0 because the integrand is odd and the interval $(-\infty,\infty)$ is symmetric with respect to the origin.

$(-1)\cdot(1) = (-1)$

odd $\cdot$ even $=$ odd

$$\int_{-\infty}^{\infty} xe^{-x^2}dx = \lim_{a\to-\infty}\int_a^0 xe^{-x^2}dx + \lim_{b\to\infty}\int_0^b xe^{-x^2}dx$$

$\boxed{\text{let } u=-x^2 \quad -\frac{1}{2}du = x\,dx}$

$$= \lim_{a\to-\infty}\frac{-1}{2}\int_{x=a}^{x=0} e^u\,du + \lim_{b\to\infty}\frac{-1}{2}\int_{x=0}^{x=b} e^u\,du$$

$$= \frac{-1}{2}\left[\lim_{a\to-\infty} e^{-x^2}\Big|_a^0 + \lim_{b\to\infty} e^{-x^2}\Big|_0^b\right]$$

$$= \frac{-1}{2}\left[(e^0 - e^{-\infty}) + (e^{-\infty} - e^0)\right] = \boxed{0}$$

11. [10 points] Determine whether the following series converges or diverges. Justify your answer.

$$\sum_{n=0}^{\infty}\frac{n3^n}{(n+1)!}$$

Ratio Test:

$$\lim_{n\to\infty}\frac{a_{n+1}}{a_n} = \lim_{n\to\infty}\frac{(n+1)3^{(n+1)}}{(n+2)!}\cdot\frac{(n+1)!}{n\,3^n}$$

$$= \lim_{n\to\infty}\frac{n+1}{n}\cdot\frac{3\cdot\cancel{3^n}}{\cancel{3^n}}\cdot\frac{\cancel{(n+1)!}}{(n+2)\cancel{(n+1)!}}$$

$$= \lim_{n\to\infty}\frac{3(n+1)}{n(n+2)} = 0 < 1$$

$\Longrightarrow$ $\boxed{\text{Converges by the Ratio Test.}}$

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12. [10 points] Find the radius of convergence for the following power series. **Do not bother to determine whether the endpoints of the interval are contained in the set of convergence.**

$$\sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!} x^n$$

Absolute Ratio Test:

$$\lim_{n\to\infty} \frac{|a_{n+1} x^{n+1}|}{|a_n x^n|} = \lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} |x|$$

$$= \lim_{n\to\infty} \left|\frac{((n+1)!)^2}{(2(n+1))!}\right| \cdot \left|\frac{(2n)!}{(n!)^2}\right| \cdot |x|$$

$$= \lim_{n\to\infty} \frac{((n+1)!)^2}{(n!)^2} \cdot \frac{(2n)!}{(2(n+1))!} \cdot |x|$$

$$\frac{((n+1)!)^2}{(n!)^2} = \frac{[(n+1)(n)(n-1)(n-2)\cdots(1)]^2}{[(n)(n-1)(n-2)\cdots(1)]^2} = \frac{(n+1)^2 \cancel{(n)^2} \cancel{(n-1)^2} \cancel{(n-2)^2} \cdots \cancel{(1)^2}}{\cancel{(n)^2} \cancel{(n-1)^2} \cancel{(n-2)^2} \cdots \cancel{(1)^2}} = (n+1)^2$$

$$\frac{(2n)!}{(2(n+1))!} = \frac{\cancel{(2n)}\cancel{(2n-1)}\cancel{(2n-2)}\cdots\cancel{(1)}}{(2n+2)(2n+1)\cancel{(2n)}\cancel{(2n-1)}\cancel{(2n-2)}\cdots\cancel{(1)}} = \frac{1}{(2n+2)(2n+1)}$$

$$\to = \lim_{n\to\infty} \frac{(n+1)^2}{(2n+2)(2n+1)} |x| = \lim_{n\to\infty} \frac{n^2+2n+1}{4n^2+6n+2} |x| = \frac{1}{4}|x| = \rho$$

We get convergence only when $\rho < 1$;

$$\frac{1}{4}|x| < 1 \implies |x| < 4 \implies \boxed{R = 4}$$

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13. [10 points] Use the following two facts:

$$\frac{d}{dx}[\sinh x] = \cosh x$$
$$\frac{d}{dx}[\cosh x] = \sinh x$$

$a = 0$ $f(x)$

to compute the $\boxed{\text{first four nonzero terms}}$ of the Maclaurin series for $\sinh x$. Leave any factorial quantities in their factorial notation: $n!$, i.e. don't bother computing a number for $n!$.

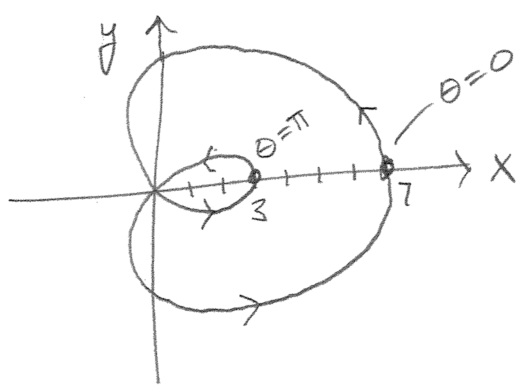
$$\boxed{f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n \qquad \text{Taylor Series formula}}$$

n	$f^{(n)}(x)$	$f^{(n)}(0)$	$\frac{f^{(n)}(0)}{n!}$
0	$\sinh x$	0	0
1	$\cosh x$	1	1
2	$\sinh x$	0	0
3	$\cosh x$	1	$\frac{1}{3!}$
4	$\sinh x$	0	0
5	$\cosh x$	1	$\frac{1}{5!}$
6	$\sinh x$	0	0
7	$\cosh x$	1	$\frac{1}{7!}$

$$\boxed{\sinh x \approx x + \frac{1}{3!}x^3 + \frac{1}{5!}x^7 + \frac{1}{7!}x^7}$$

14. [10 points] Sketch the graph of the following polar function.

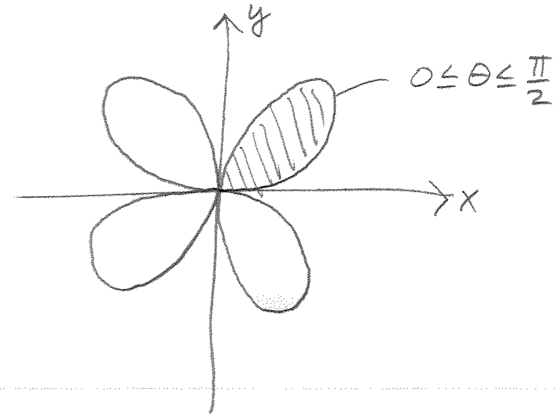
$$r = 2 + 5\cos\theta \qquad 0 \le \theta \le 2\pi$$



15. [10 points] Find the area of the first petal of the four–leaved rose determined by

$$r = 4\sin 2\theta \qquad 0 \le \theta \le \pi/2.$$

$$\boxed{A = \frac{1}{2}\int_{\alpha}^{\beta} [f(\theta)]^2\,d\theta \quad \text{where } r = f(\theta)}$$



$$A = \frac{1}{2}\int_0^{\pi/2} 16\sin^2(2\theta)\,d\theta$$

$$= 8\int_0^{\pi/2} \sin^2(2\theta)\,d\theta$$

$$\boxed{\begin{array}{l}\underline{\text{Double angle formula:}}\\ \sin^2(\theta) = \frac{1}{2}(1-\cos(2\theta))\end{array}}$$

$$= 8\int_0^{\pi/2} \frac{1}{2}(1-\cos(4\theta))\,d\theta$$

$$\boxed{u = 4\theta \qquad \frac{1}{4}du = d\theta}$$

$$= 4\left[\int_0^{\pi/2} d\theta - \frac{1}{4}\int_{\theta=0}^{\theta=\pi/2} \cos(u)\,du\right]$$

$$= 4\left[\theta\Big|_0^{\pi/2} - \frac{1}{4}\cancel{\sin(4\theta)}^{\,0}\Big|_0^{\pi/2}\right]$$

$$= 4\cdot\frac{\pi}{2} = \boxed{2\pi}$$

✓