§ 9.8 Taylor and Maclaurin Series

We've looked at ways of manipulating power series to get new power series. Now we ask the obvious question:

Q: Can *any* function be represented by a power series?

A: (forthcoming)

Let's assume we can represent any function by a power series centered on a, i.e. $f(x) = a_0 + a_1(x-a) + a_2(x-a)^2 + \cdots$

What do higher derivatives of $f(x)$ say about the a_n's?

$$f'(x) = a_1 + 2a_2(x-a) + 3a_3(x-a)^2 + 4a_4(x-a)^3 + \cdots$$

$$f''(x) = 2a_2 + 3\cdot 2a_3(x-a) + 4\cdot 3\, a_4(x-a)^2 + 5\cdot 4 a_5(x-a)^3 + \cdots$$

$$f'''(x) = 3\cdot 2 a_3 + 4\cdot 3\cdot 2\, a_4(x-a) + 5\cdot 4\cdot 3\, a_5(x-a)^2 + 6\cdot 5\cdot 4\, a_6(x-a)^3 + \cdots$$

Let $\boxed{x=a}$ and solve for a_n in each equation above

$$f'(a) = a_1$$

$$f''(a) = 2a_2$$

$$f'''(a) = 3\cdot 2\, a_3$$

$$f^{(4)}(a) = 4\cdot 3\cdot 2\, a_4$$

$$\vdots$$

$$f^{(n)}(a) = n!\, a_n$$

$$\boxed{a_n = \frac{f^{(n)}(a)}{n!}}$$

The above analysis tells us two things:

1. The coefficients, a_n depend on f.
2. The coefficients are <u>unique</u> for each f.

Def A <u>Maclaurin series</u> is a power series representation of a function centered around 0.

Def A <u>Taylor Series</u> is a power series representation of a function centered around a constant "a".

Ex <u>Taylor Series</u> $f(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-3)^3 + \cdots$

<u>Maclaurin Series</u> $f(x) = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4 + \cdots$

So a Maclaurin Series is just a special case of a Taylor series with the constant $a = 0$.

Ex What is the Maclaurin series of $f(x) = 3x^2 + 2x + 1$?

Recall : $c_n = \dfrac{f^{(n)}(a)}{n!}$ here $a = 0$.

n	$f^{(n)}(x)$	$f^{(n)}(0)$	$c_n = \dfrac{f^{(n)}(0)}{n!}$
0	$3x^2+2x+1$	1	$\frac{1}{0!} = 1$
1	$6x+2$	2	$\frac{2}{1!} = 2$
2	6	6	$\frac{6}{2!} = 3$
3	0	0	0

$\Rightarrow f(x) = 1 + 2x + 3x^2 + 0x^3 + 0x^4 + \cdots$

$= 1 + 2x + 3x^2$

$\Rightarrow$ So the idea agrees with polynomial functions.

Ex What is the Taylor Series of $f(x) = 3x^2 + 2x + 1$ centered at $x = 2$?

n	$f^{(n)}(x)$	$f^{(n)}(2)$	$c_n = \frac{f^{(n)}(2)}{n!}$
0	$3x^2+2x+1$	$3\cdot 2^2 + 2\cdot 2 + 1 = 17$	$\frac{17}{0!} = 17$
1	$6x+2$	$6\cdot 2 + 2 = 14$	$\frac{14}{1!} = 14$
2	6	6	$\frac{6}{2!} = 3$
3	0	0	0

$$f(x) = 17 + 14(x-2) + 3(x-2)^2 + 0(x-2)^3 + \cdots$$

Is this really the same function? Let's expand...

$$f(x) = 17 + 14x - 28 + 3(x^2 - 4x + 4)$$
$$= 17 - 28 + \underline{14x} + 3x^2 \underline{-12x} + 12$$
$$= 3x^2 + 2x + 1 \quad \checkmark$$

They do match!

Thm A Uniqueness Theorem

If $f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n$ has radius of convergence $R > 0$,

Then $c_n = \dfrac{f^{(n)}(a)}{n!}$.

Remark Having non-zero radius of convergence implies $f(x)$ is differentiable and we can differentiate $f(x)$ infinitely many times.

Ex Find the Maclaurin series for $f(x) = e^x$.

Solution Maclaurin means $a = 0$ in $f(x) = \sum_{n=0}^{\infty} \underbrace{\frac{f^{(n)}(a)}{n!}}_{a_n} (x-a)^n$

$f(x) = e^x$

n	$f^{(n)}(x)$	$f^{(n)}(0)$	$\frac{f^{(n)}(0)}{n!} = a_n$
0	e^x	$e^0 = 1$	$\frac{1}{0!} = 1$
1	e^x	$e^0 = 1$	$\frac{1}{1!} = 1$
2	e^x	$e^0 = 1$	$\frac{1}{2!}$
3	e^x	$e^0 = 1$	$\frac{1}{3!}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
n	e^x	$e^0 = 1$	$\frac{1}{n!}$

$$\Rightarrow \boxed{e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \cdots}$$

What is the radius of convergence, R? Use Absolute Ratio Test:

$$\lim_{n\to\infty} \left|\frac{x^{n+1}}{(n+1)!}\right| \cdot \left|\frac{n!}{x^n}\right| = \lim_{n\to\infty} \left|\frac{\cancel{x^n} \cdot x}{\cancel{x^n}}\right| \frac{\cancel{n!}}{(n+1)\cancel{n!}} = \lim_{n\to\infty} \frac{|x|}{n+1} = 0$$

So $\rho = 0$ for all real x $\Rightarrow R = \frac{1}{\rho} = \frac{1}{0} = \boxed{\infty}$

$\Rightarrow \sum_{n=0}^{\infty} \frac{x^n}{n!}$ converges absolutely for all real x!

Ex Find the Taylor Series for $f(x) = \sin x$ centered on $a = 2\pi$.

Solution

n	$f^{(n)}(x)$	$f^{(n)}(2\pi)$	$\frac{f^{(n)}(2\pi)}{n!}$
0	$\sin(x)$	0	0
1	$\cos(x)$	1	$\frac{1}{1!}$
2	$-\sin x$	0	0
3	$-\cos x$	-1	$\frac{-1}{3!}$
4	$\sin x$	0	0 ← pattern repeats

$$\Rightarrow \boxed{\sin(x) = (x-2\pi) - \frac{1}{3!}(x-2\pi)^3 + \frac{1}{5!}(x-2\pi)^5 - \frac{1}{7!}(x-2\pi)^7 + \cdots}$$

$$\boxed{\sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}(x-2\pi)^{2n+1}}$$

Radius of convergence, R ?

Remark: The radius of convergence will be the same regardless of whether we expand around $a = 2\pi$ or $a = 0$ (Maclaurin), thus to make calculations easier set $a = 0$.

$$\lim_{n\to\infty} \left|\frac{x^{2(n+1)+1}}{(2(n+1)+1)!}\right| \cdot \left|\frac{(2n+1)!}{x^{2n+1}}\right| = \lim_{n\to\infty} \frac{\cancel{(2n+1)!}}{(2n+3)(2n+2)\cancel{(2n+1)!}} |x^2| = 0$$

Note: $2(n+1)+1 = 2n+3$ $\qquad \rho = 0 \Rightarrow R = \frac{1}{\rho} = \boxed{\infty}$

Check the next page to see graphs of $\sum_{n=0}^{N} \frac{(-1)^n}{(2n+1)!} x^{2n+1}$ $\quad N = 1, \ldots, 13 \rightarrow$

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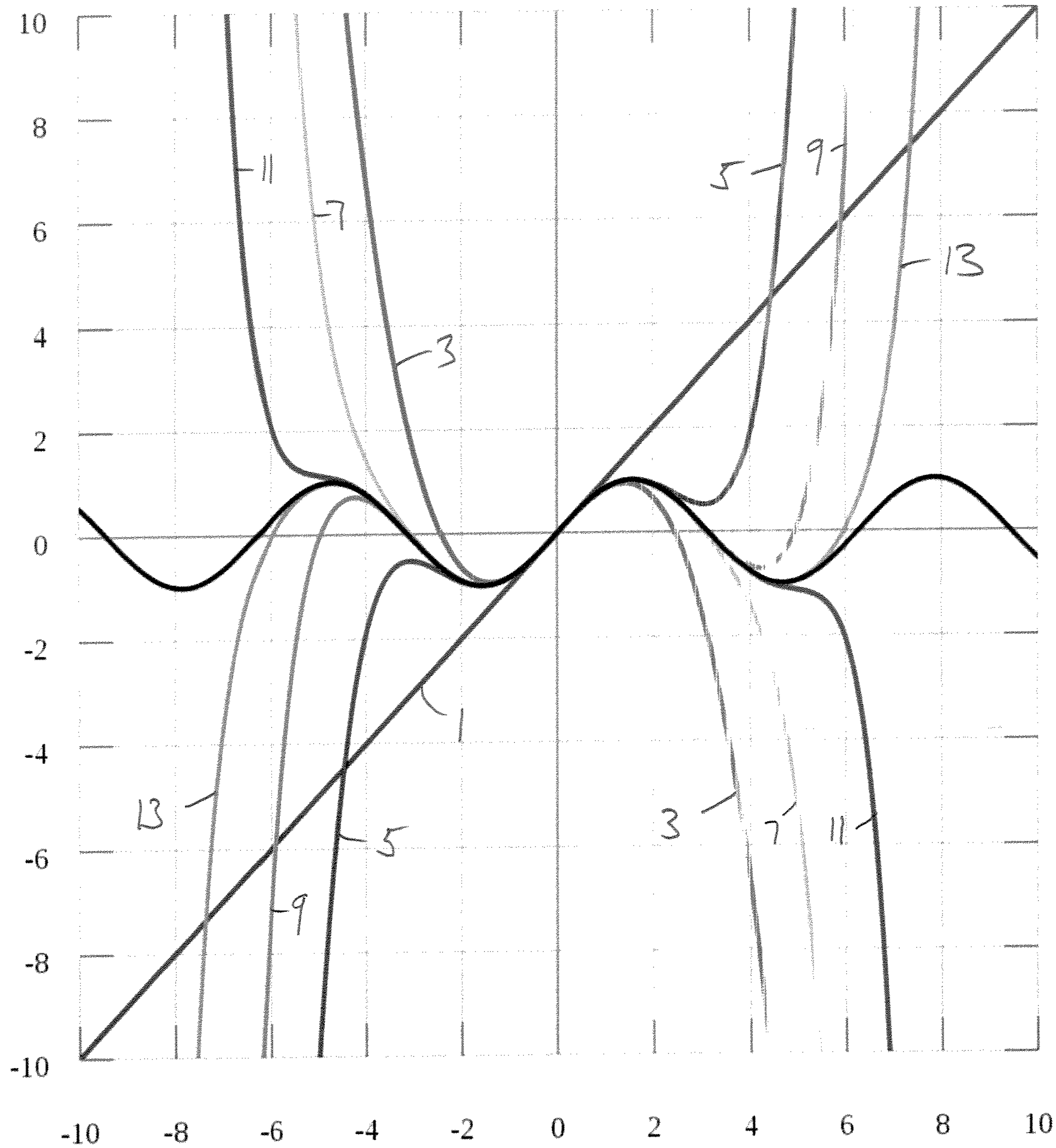

The above plot shows the graph of $\sin(x)$ and graphs of partial sums of $\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}$)

that is it plots $f(x) = \sum_{n=0}^{N} \frac{(-1)^n}{(2n+1)!} x^{2n+1}$ for

$N = 1, 3, 5, 7, 9, 11, 13$

Ex. We can differentiate the Taylor series for sin(x) centered on 2π to get a Taylor series for cos(x) @ 2π!

$$\frac{d}{dx}\left[\sin(x) = (x-2\pi) - \frac{1}{3!}(x-2\pi)^3 + \frac{1}{5!}(x-2\pi)^5 - \frac{1}{7!}(x-2\pi)^7 + \cdots\right]$$

$$\cos(x) = 1 - \frac{3}{3!}(x-2\pi)^2 + \frac{5}{5!}(x-2\pi)^4 - \frac{7}{7!}(x-2\pi)^6 + \cdots$$

$$\cos(x) = 1 - \frac{1}{2!}(x-2\pi)^2 + \frac{1}{4!}(x-2\pi)^4 - \frac{1}{6!}(x-2\pi)^6 + \cdots$$

Radius of convergence? Recall that differentiating or integrating a power series term by term does not change the radius of convergence, i.e. the new series has the same radius of convergence as the previous one.

R for sin(x) was $\infty \Longrightarrow$ R for cos(x) is ∞ too!

<u>Note</u> We could have expanded sin(x) around any integral multiple of π, say $n\pi$, then if n is even we get the same exact results as above. If n is odd, then this is equivalent to multiplying every term by (-1). So usually we just give the power expansion of trig functions around 0:

$$\boxed{\begin{aligned} \sin x &= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{(2n+1)} = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \cdots \\ \cos x &= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{(2n)} = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \cdots \\ e^x &= \sum_{n=0}^{\infty} \frac{1}{n!}x^n = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \cdots \end{aligned}}$$

<u>Def</u> The <u>Taylor polynomial of order n</u> based at a is

$$P_n(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \dots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

$$\boxed{P_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k}$$

<u>Def</u> The n^{th} Taylor remainder is

$$\boxed{R_n(x) = f(x) - P_n(x)}$$

<u>Thm</u> A function equals its Taylor series at x if $\lim_{n\to\infty} R_n(x) = 0$.

<u>Note</u>: Your text defines $R_n(x)$ slightly differently so as to make the proof of the above theorem easier. The definition I gave above is more natural and intuitive.

Now we can finally answer the question at the beginning:

<u>Q</u>: Can any function be represented by a power series?

<u>A</u>: No, only if $\lim_{n\to\infty} R_n(x) = 0$

<u>Ex.</u> If $f(x) = \begin{cases} e^{-1/x^2} & x \neq 0 \\ 0 & x = 0 \end{cases}$ then $R_n(x) \not\to 0$ as $n \to \infty$.

However many, many functions can be faithfully represented.