§9.7 Operations on Power Series

Recall that found that if we take an ordinary power series, say $\sum_{n=0}^{\infty} a_n x^n$ and replace the a_n's with a constant say 1, that is, $a_n = 1$ for all n, then:

$$\sum_{n=0}^{\infty} a x^n = \frac{a}{1-x} \implies \underset{(\text{for all } n)}{\text{set } a_n = 1} \quad \boxed{\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}}$$

We can manipulate this series in several ways to get new series. First recall:

① The radius of convergence of $\frac{1}{1-x}$ is 1, so

② the radius " " " $\frac{1}{1-x/a}$ is a.

(Because $\frac{x}{a} < 1 \iff x < a$) $\quad \boxed{\sum_{n=0}^{\infty} \left(\frac{x}{a}\right)^n = \frac{1}{1-\frac{x}{a}}}$

Thm Power Series Can Be Differentiated and Integrated Term by Term

If $f(x) = \sum_{n=0}^{\infty} a_n x^n$ has a non-zero radius of convergence R, then

① $f'(x) = \sum_{n=1}^{\infty} a_n \cdot n x^{n-1}$, with radius of convergence R.

② $\int f(x)\,dx = \sum_{n=-1}^{\infty} \frac{a_n}{n+1} x^{n+1}$, with radius of convergence R.

Ex Differentiate Term by Term

$$\boxed{\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots}$$

$$\Rightarrow \frac{d}{dx}\left(\frac{1}{1-x}\right) = \frac{d}{dx}(1) + \frac{d}{dx}(x) + \frac{d}{dx}(x^2) + \frac{d}{dx}(x^3) + \cdots$$

$$\frac{1}{(1-x)^2} = 0 + 1 + 2x + 3x^2 + \cdots \qquad -1 < x < 1$$

$$\Rightarrow \boxed{\frac{1}{(1-x)^2} = \sum_{n=0}^{\infty} (n+1)x^n \qquad -1 < x < 1}$$

Integrate Term by Term

$$\int \frac{1}{1-x}\,dx = \int 1\,dx + \int x\,dx + \int x^2\,dx + \int x^3\,dx + \cdots$$

$u = 1-x \quad du = -dx \quad -du = dx$

$$\int \frac{-1}{u}\,du = -\ln(1-x) = x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \cdots \qquad -1 < x < 1$$

Replace x with $-x$ and multiply both sides by -1:

$$\ln(x+1) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots \qquad -1 < x < 1$$

$$\Rightarrow \boxed{\ln(x+1) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1} \qquad -1 < x < 1}$$

<u>Ex</u> $\quad \dfrac{1}{1+x} = \dfrac{1}{1-(-x)} = \displaystyle\sum_{n=0}^{\infty} (-1)^n x^n \qquad -1<x<1$

So $\quad \ln(1+x) = \displaystyle\int \frac{1}{1+x}\,dx = \sum_{n=0}^{\infty} \int (-1)^n x^n\,dx$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1} \qquad -1<x<1$$

which is the same answer we just got.

Have to check what happens at $|x|=1$:

<u>$x=1$</u> $\quad \displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{n+1}$ which $\boxed{\text{converges by Alt. Series Test.}}$

Also $\ln(1+1) = \ln(2) = \sum \dfrac{(-1)^n}{n+1}$

no exponent necessary because $2n+1$ is odd.

<u>$x=-1$</u> $\quad \displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{n+1}(-1)^{n+1} = \sum_{n=0}^{\infty} \frac{(-1)^{2n+1}}{n+1} = \sum_{n=0}^{\infty} \frac{(-1)}{n+1} = -\sum_{n=0}^{\infty} \frac{1}{n+1}$

$\boxed{\text{diverges by limit comparison test w/ harmonic series}}$

$\Longrightarrow \quad \boxed{-1 < x \leq 1}$ convergence set for

Ex. We found the integration formula $\int \frac{1}{1+x^2}\,dx = \tan^{-1}x + C$ in section 6.8 and again in section 7.4 by different means. Now we show a third way to compute $\tan^{-1}x$ (the way your calculator does it).

Solution

Recall $\quad \frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots \quad (-1<x<1)$

Replace "x" with "$-x^2$":

$$\frac{1}{1+x^2} = \sum_{n=0}^{\infty}(-x^2)^n = \sum_{n=0}^{\infty}(-1)^n x^{2n} = \overset{n=0}{1} \overset{n=1}{- x^2} \overset{n=2}{+ x^4} \overset{n=3}{- x^6} + \cdots$$

$$\Rightarrow \int \frac{1}{1+x^2}\,dx = \int\left(\sum_{n=0}^{\infty}(-1)^n x^{2n}\right)dx = \sum_{n=0}^{\infty}(-1)^n\left(\int x^{2n}\,dx\right)$$

This step is where we use the theorem on page 1.

$$= \int 1\cdot dx - \int x^2\,dx + \int x^4\,dx - \int x^6\,dx + \cdots$$

$$= x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots$$

$$\Rightarrow \boxed{\int \frac{1}{1+x^2}\,dx = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots = \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{2n+1}}$$

Note that in both endpoint cases $x=1$, $x=-1$ we get an alternating series of decreasing terms, so the power series converges for $x=1$, $x=-1$ and we get

$\boxed{-1 \le x \le 1}$ as the convergence set.

Ex Another Reason why power series are so important in mathematics is because they can be used to solve many difficult differential equations. Here we solve an easy one, namely $\boxed{y' = ky}$ w/ initial condition $\boxed{y(0) = y_0}$ (a constant). Recall how we solved this in 6.6.

Solution $y' = ky \Rightarrow \frac{dy}{dx} = ky \Rightarrow \int \frac{dy}{y} = \int k\,dx$

$\Rightarrow \ln|y| = kx + C \Rightarrow |y| = e^{kx}e^{C} \Rightarrow \boxed{y(x) = y_0 e^{kx}}$

Power Series Solution

1. Assume that the solution function to $y' = ky$ is expressible as a power series in x, then:

$$\boxed{ky(x) = k\sum_{n=0}^{\infty} a_n x^n = k\left[a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots\right] \quad (*)}$$

Solving the ordinary differential equation (ODE) boils down to figuring out a formula for a_n!

2. Differentiate $(*)$ term by term:

$$\boxed{y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} = a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + \cdots \quad (**)}$$

3. Equate like coefficients in $(*)$ and $(**)$:

$$\begin{array}{ll} ka_0 = a_1 & ka_3 = 4a_4 \\ ka_1 = 2a_2 & ka_4 = 5a_5 \\ ka_2 = 3a_3 & \vdots \\ & ka_n = (n+1)a_{n+1} \end{array} \Rightarrow \boxed{a_{n+1} = k\frac{a_n}{n+1}} \rightarrow$$

recurrence relation

4 Our next goal now that we have the recurrence relation $\boxed{a_{n+1} = k\left(\frac{a_n}{n+1}\right)}$ is to use it to find a formula for a_n that does not depend on previous terms (i.e. a_{n-1}):

$$a_1 = k a_0$$

$$a_2 = k \frac{a_1}{2} = k \frac{k a_0}{2} = k^2 \frac{a_0}{2!}$$

$$a_3 = k \frac{a_2}{3} = k \cdot \frac{k^2 a_0}{3 \cdot 2!} = k^3 \frac{a_0}{3!}$$

$$a_4 = k \frac{a_3}{4} = k \cdot \frac{k^3 a_0}{4 \cdot 3!} = k^4 \frac{a_0}{4!}$$

$$\vdots$$

$$\boxed{a_n = k^n \frac{a_0}{n!}}$$

Note $0! = 1$ (by definition)

Now plug our formula for a_n back into $(*)$:

$$y(x) = \sum_{n=0}^{\infty} \left(k^n \frac{a_0}{n!}\right) x^n = a_0 \sum_{n=0}^{\infty} \frac{(kx)^n}{n!}$$

$$y(x) = \underbrace{a_0}_{y_0} \underbrace{\left(1 + kx + \frac{(kx)^2}{2!} + \frac{(kx)^3}{3!} + \frac{(kx)^4}{4!} + \dots\right)}_{e^{kx}}$$

We get the same solution to the ODE, and we got a power series formula for $\boxed{e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots}$