§ 9.6 Power Series

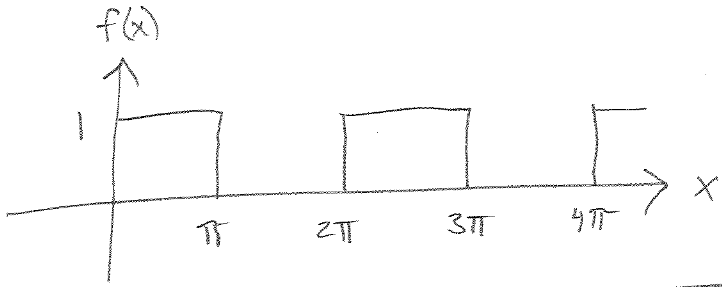
We are now ready to make the conceptual leap from infinite series of real numbers to infinite series of functions of a real variable.

Ex. Fourier Series

A Fourier Series is simply an infinite series where the functions of x being summed are sines and cosines:

$$f(x) = \frac{1}{2} + \sum_{n=1}^{\infty} \frac{1-(-1)^n}{n\pi} \sin(nx)$$

When you graph the above Fourier series you get a square wave:



We will not study Fourier series in this course. We will study a simpler case called power series, where the functions of x are just positive integral powers of x, or $(x-a)$.

example

$$f(x) = \sum_{n=0}^{\infty} a_n x^n = \underbrace{a_0 x^0}_{= a_0 \cdot 1} + a_1 x^1 + a_2 x^2 + a_3 x^3 + \cdots$$

note we usually start at zero now.

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Def A power series in x is a function of x.

The natural question to ask when confronted with a given power series is: "For what values of x does the power series converge?".

Ex. What if $a_n = a$ for all n (i.e. all the coefficients in the power series are the same constant)?

$$\sum_{n=0}^{\infty} ax^n = a + ax + ax^2 + ax^3 + ax^4 + \cdots$$

Notice that if we reindex by setting $\boxed{n = k-1}$, we get the geometric series:

$$\sum_{n=0}^{\infty} ax^n \xrightarrow[n \mapsto k-1]{} \sum_{k-1=0}^{\infty} ax^{k-1} = \sum_{k=1}^{\infty} ax^{k-1} = \frac{a}{1-x}$$

which only holds for $-1 < x < 1$, i.e. $|x| < 1$, so the power series $\sum_{n=0}^{\infty} ax^n$ converges on the set $\boxed{-1 < x < 1}$.

convergence set

Thm Convergence set of a power series

The convergence set for a power series $\sum_{n=0}^{\infty} a_n x^n$ is always an interval of one of the three types:

1) The single point $x = 0$.
2) An interval $(-R, R)$, plus possibly one or both endpoints.
3) The whole real line, $(-\infty, \infty)$.

The three cases are said to have radius of convergence, $0, R, \infty$.

Ex what is the convergence set for $\sum_{n=0}^{\infty} \frac{x^n}{(n+1)2^n}$

Solution: Use the Absolute Ratio Test:

$$\lim_{n\to\infty} \frac{|x^{n+1}|}{(n+2)2^{n+1}} \cdot \frac{(n+1)2^n}{|x^n|} = \lim_{n\to\infty} \frac{|x|}{2} \cdot \frac{(n+1)}{(n+2)} = \frac{|x|}{2} = \rho$$

Recall that we get convergence when $\rho < 1$, thus

$$\frac{|x|}{2} < 1 \iff |x| < 2 \iff \boxed{-2 < x < 2}.$$

Also recall that the test is inconclusive when $\rho = 1$, i.e.

when $x = 2$ or $x = -2$, so we examine these cases separately:

$x=2$ $\quad \sum_{n=0}^{\infty} \frac{\cancel{2^n}}{(n+1)\cancel{2^n}} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$ the harmonic series (which diverges)

$x=-2$ $\quad \sum_{n=0}^{\infty} (-1)^n \frac{\cancel{2^n}}{(n+1)\cancel{2^n}} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$ the alternating harmonic series (which converges conditionally)

$$\Longrightarrow \boxed{-2 \le x < 2} \text{ is the convergence set.}$$

Ex. Find the radius of convergence of $\sum_{n=0}^{\infty} \frac{n}{3^n} x^n$

Solution: Use the absolute ratio test:

$$\lim_{n\to\infty} \left| \frac{(n+1)x^{n+1}}{3^{n+1}} \right| \cdot \left| \frac{3^n}{n x^n} \right| = \lim_{n\to\infty} \frac{n+1}{n} \cdot \frac{\cancel{3^n}}{3 \cdot \cancel{3^n}} \cdot \frac{|x \cdot \cancel{x^n}|}{|\cancel{x^n}|} = \frac{|x|}{3} = \rho$$

Thus when $\rho < 1$ the power series will converge $\Rightarrow \boxed{|x| < 3}$

Test the end-points $x=3$ and $x=-3$. Plug in and apply any test

$x=3$ $\quad \sum_{n=0}^{\infty} \frac{n}{3^n} x^n \longrightarrow \sum_{n=0}^{\infty} \frac{n}{\cancel{3^n}} \cancel{3^n} = \sum_{n=0}^{\infty} n$ diverges $\Rightarrow$ don't include $x=3$.

Goal for next three sections:

$\boxed{\text{Find power series expansions of functions.}}$

We know $\boxed{\dfrac{a}{1-x} = \sum_{n=0}^{\infty} a x^n} \Longrightarrow \boxed{\dfrac{1}{1-x} = \sum_{n=0}^{\infty} x^n}$

Our first and most important function!

1) Replace "x" in $\frac{1}{1-x}$ with "x^2":

$$\frac{1}{1-x^2} = \sum_{n=0}^{\infty} (x^2)^n = \sum_{n=0}^{\infty} x^{2n} = 1 + x^2 + x^4 + x^6 + \ldots$$

2) Replace "x" in $\frac{1}{1-x}$ with "$-x^2$":

$$\frac{1}{1+x^2} = \sum_{n=0}^{\infty} (-x^2)^n = \sum_{n=0}^{\infty} (-1)^n x^{2n} = 1 - x^2 + x^4 - x^6 + \ldots$$

3) Replace "x" in $\frac{1}{1-x}$ with "$x-1$":

$$\frac{1}{2-x} = \frac{1}{1-(x-1)} = \sum_{n=0}^{\infty} (x-1)^n = 1 + (x-1) + (x-1)^2 + (x-1)^3 + \ldots$$

Ex Find a power series representing: $\frac{x}{1-x^2}$

$$\frac{\overset{a_0}{x}}{1-\underset{r}{x^2}} = x \cdot 1 + x \cdot (x^2)^1 + x \cdot (x^2)^2 + x \cdot (x^2)^3 + \ldots$$

$$= x + x^3 + x^5 + x^7 + \ldots$$

$$= \boxed{\sum_{n=0}^{\infty} x^{(2n+1)}}$$