§9.5 Alternating Series, Absolute Convergence, and Conditional Convergence

Def An alternating series is a series in which the sign of the terms changes or alternates between '+' and '−' from one term to the next. This is usually written with $(-1)^{n+1}$ or $(-1)^n$ in "Σ" notation.

Ex

$$1 - \frac{1}{2} + \frac{1}{3} + \frac{-1}{4} + \frac{1}{5} + \frac{-1}{6} + \cdots = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n}$$

This series is called the alternating harmonic series.

$$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n} = \ln 2 \approx .69$$

Thus the alternating harmonic series converges to $\ln 2$. We will defer the proof of this fact until we study Taylor Series.

Thm Alternating Series Test (important)

Let $a_1 - a_2 + a_3 - a_4 + a_5 - \cdots$ be an alternating series.

If ($a_n > a_{n+1}$ for all n and $\lim_{n\to\infty} a_n = 0$)

i.e. the sequence $|a_n|$ is decreasing

Then the alternating series converges.

Ex. The alternating harmonic series converges.

$$\left(\frac{1}{n} > \frac{1}{n+1} \text{ for all } n\right) \text{ and } \left(\lim_{n\to\infty} \frac{1}{n} = 0\right)$$

$\Rightarrow$ converges by the alternating series test.

Ex.

a) $1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots$ alternating p-series

$a_n = (-1)^{n+1}\frac{1}{n^2}$ disregarding signs, $\left(a_n = \frac{1}{n^2} > \frac{1}{(n+1)^2} = a_{n+1}\right.$ for all $\left.n\right)$ AND $\left(\lim_{n\to\infty} \frac{1}{n^2} = 0\right) \Rightarrow$ $\boxed{\text{converges by Alt. Ser. Test}}$

b) $\frac{1}{2} - \frac{1}{5} + \frac{1}{10} - \frac{1}{17} + \cdots = \sum_{n=1}^{\infty} (-1)^{n+1}\frac{1}{n^2+1}$

$\left(a_n = \frac{1}{n^2+1} > \frac{1}{(n+1)^2+1} = a_{n+1}\right.$ for all $\left.n\right)$ AND $\left(\lim_{n\to\infty} \frac{1}{n^2+1} = 0\right)$

$\Rightarrow$ $\boxed{\text{converges by the Alt. Series Test}}$

c) $\frac{1}{e} - \frac{2}{e^2} + \frac{3}{e^3} - \frac{4}{e^4} + \cdots = \sum_{n=1}^{\infty} (-1)^{n+1}\frac{n}{e^n}$

$\left(a_n = \frac{n}{e^n} > \frac{n+1}{e^{n+1}} = a_{n+1}\right.$ for all $\left.n\right)$ AND $\left(\lim_{n\to\infty} \frac{n}{e^n} = 0\right)$

$\Rightarrow$ $\boxed{\text{converges by the Alt. Series Test}}$

Thm Absolute Convergence Test (important)

If $\sum |a_n|$ converges then $\sum a_n$ converges.

Ex $1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \cdots = \sum_{n=1}^{\infty} (-1)^{n+1} \left(\frac{1}{2}\right)^{n-1}$

If we change all signs to positive we get $\sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^{n-1}$ which is a geometric series with $r = \frac{1}{2}$ and thus converges.

Therefore $\sum_{n=1}^{\infty} (-1)^{n+1} \left(\frac{1}{2}\right)^{n-1}$ converges by the Absolute Con. Test.

Def A series $\sum_n a_n = a_1 + a_2 + a_3 + \cdots$ is said to be <u>absolutely convergent</u> if $\sum |a_n| = |a_1| + |a_2| + |a_3| + \cdots$ converges.

Def A series $\sum a_n = a_1 + a_2 + a_3 + \cdots$ is said to be <u>conditionally convergent</u> if $\sum a_n$ converges, but $\sum |a_n|$ diverges.

Q: Is the alternating harmonic series absolutely convergent?

A: No, but it is conditionally convergent.

These definitions partition the set of infinite series into three disjoint classes.

absolutely convergent series

conditionally convergent series

divergent series

Thm Absolute Ratio Test (very important)

Let $\sum a_n$ be a series of nonzero terms, where

$$\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \rho \quad \text{"rho"}$$

1) If $\rho < 1$ then $\sum a_n$ converges absolutely
2) If $\rho > 1$ then $\sum a_n$ diverges.
3) If $\rho = 1$ then the test is inconclusive

Note: This test is very useful when a_n involves factorials.

Ex $\sum_{n=1}^{\infty} \frac{1}{n!}$ We showed this converges by the Bounded Sum Test. But now we can prove absolute convergence. (See Notes for 9.3)

Solution

$$\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \frac{\left|\frac{1}{(n+1)!}\right|}{\left|\frac{1}{n!}\right|} = \lim_{n\to\infty} \frac{n!}{(n+1)!}$$

$$= \lim_{n\to\infty} \frac{\cancel{n(n-1)(n-2)\cdots(3)(2)(1)}}{(n+1)\cancel{n(n-1)(n-2)\cdots(3)(2)(1)}}$$

$$= \lim_{n\to\infty} \frac{1}{n+1} = 0 = \rho$$

Since $\rho < 1 \implies$ $\boxed{\sum \frac{1}{n!} \text{ converges absolutely by the absolute ratio test.}}$

key fact to remember:

$(n+1)! = (n+1)n!$

because: $(n+1)! = (n+1)[(n)(n-1)(n-2)\cdots(3)(2)(1)] = (n+1)\,n!$

Ex $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{3^n}{n!}$

$$\lim_{n\to\infty} \frac{|3^{n+1}|}{|(n+1)!|} \cdot \frac{|n!|}{|3^n|} = \lim_{n\to\infty} \frac{\cancel{n!}\, 3 \cdot \cancel{3^n}}{(n+1)\cancel{!}\, \cancel{3^n}} = \lim_{n\to\infty} \frac{3}{n+1} = 0 = \rho$$

$\rho < 1 \Rightarrow$ $\boxed{\sum_{n=1}^{\infty} (-1)^{n+1} \frac{3^n}{n!} \text{ converges absolutely by the absolute ratio test}}$

Ex, $\sum (-1)^{n+1} \frac{1}{(n!)^3}$ abs. conv.

$$\lim_{n\to\infty} \frac{1}{((n+1)!)^3} \cdot \frac{(n!)^3}{1} = \lim_{n\to\infty} \frac{\cancel{(n!)^3}}{(n+1)(n+1)(n+1)\cancel{(n!)^3}}$$

$$= \lim_{n\to\infty} \frac{1}{(n+1)^3} = 0 = \rho < 1$$

$\boxed{\text{Thus } \sum (-1)^{n+1} \frac{1}{(n!)^3} \text{ converges absolutely by the absolute ratio test.}}$

Two Interesting Facts:

> **Thm** Rearrangement Theorem
>
> The terms of an absolutely convergent series can be rearranged without affecting either the convergence or the sum of the series.

That makes intuitive sense because that is exactly how finite sums behave. It is natural to think that the same is true for conditionally convergent series, but it's not!

> **Thm** [Riemann]
>
> The terms of a conditionally convergent series can be rearranged so that the series sums to <u>any</u> real number.

Pretty amazing eh?! The above thm is actually not too hard to prove either. A proof of the second theorem is usually done in Math 3210.

Riemann's theorem is simply a logical consequence of defining the sum of an infinite series to be the limit of its sequence of partial sums. The ability to rearrange the series allows one to "monkey" with the sequence of partial sums and thus change its limit. Note that we could have chosen to define the sum of an infinite series in some other way. But this is Calculus where the "limit" is our tool.